An Age Dependent Semi-Markov Model of Marital Status in Belgium : An Application of Littman's Algorithm to Period Data, 1970.

Fernando RAJULTON

IPD-Working Paper 1984-1

Acknowledgement :

Thanks are due to Stan Wijewickrema for his patient perusal of the draft and for valuable suggestions for clarifying certain points invariably taken for granted by a writer.

ORDERING INFORMATION

Extra copies of this working paper are available from :

Mr. E. Vanden Balck
Secretary/Librarian
Interuniversity Programme in Demography
c/o Centrum voor Sociologie
Vrije Universiteit Brussel
Pleinlaan 2
B-1050 Brussels
BELGIUM

at the cost of US $ 2.50 or BF 130. Please attach check or international money order.

ORDER FORM

Name ....

Address ....

....

....

....

....

wishes to receive ...... copies of Working Paper Nr..........

and includes payment of US $ .... or BF........

ABSTRACT

Of a few suggestions put forward to relax the Markovian assumption inherent in the multistate life tables currently in use, that of Charles J.Mode is found to be the most helpful. An age-dependent semi-Markov model from the sample path perspective as suggested by Mode makes feasible a computer algorithm. This algorithm (which incorporates the Littman algorithm) enables a more relevant and a more realistic analysis of transitions between states through first passage probabilities and renewal densities, in terms of duration spent in various states and in terms of "pulls and pushes" among states. Further, the first passage probabilities lend themselves to parametrization which is of great help in further studies of effects of heterogeneities in the population. The model is applied to period data (1970) of marital states in Belgium and its implications are pointed out with an illustrative example. In particular, the Hernes' model applied to the first passage probabilities renders interesting interpretations of sociological forces in operation behind the transitions between marital states.

CONTENTS

P A R T I

1. INTRODUCTION

The analytic power of the multistate demographic models rests on the basic assumptions of homogeneity and Markovian behaviour. These two assumptions imply that all the individuals of a given age present at the same time in a given state have identical propensities for moving out of that state (the homogeneity assumption) and that these propensities are independent of the past history of the individuals (the Markovian assumption).

However much the analytic power may have been enhanced by these Markov-based models in demographic analysis, they are still unrealistic in portraying the obvious heterogeneous world phenomena. Some attempts have been made in relaxing these assumptions in some way or other, but mainly within the Markovian set-up. Thus, for example, Ledent (1980) suggest the possibility of reducing the effects of the homogeneity assumption by introducing place-of-birth specifications in the construction of multiregional life tables; through which a population, instead of being analysed as a single homogeneous entity, is divided into a few homogeneous groups. Kitsul and Philipov(1981) suggest the high-and-low intensity movers model (based on the classic mover-stayer model) in the context of reconciling demographic data collected over different periods of time. Such attempts carry on the demographic tradition of age-dependence in rates, in spite of the recognition of the effect of duration in demographic analysis, be it in the context of single state or multistate analysis.

If the duration variable were to be included in the analysis, it would have the implication that moves between states are dependent on the length of stay in the state of origin. This dependence on the length of stay in a state cannot be studied through these Markov-based models. This is not only because of the Markovian assumption which forgets the history of the individuals, but also because of the forward Kolmogorov differential equations on which these models have been constructed. Analytically, the forward equations consider only the last jump in a series of moves and "forget" how long an individual has stayed in a particular state before making this jump. In other words, whatever be the sojourn time in a particular state, the probability of making a jump is exponentially distributed, and hence is duration independent. In many phenomena considered in demography or in the other social sciences, sojourn times with exponential distributions would not fit the facts, as duration in a state does affect the probability of moving out of that state, especially when age effects are known to be important.

To accomodate the effects of duration and other inhomogeneities along with the age effect, a semi-Markov model has long been suggested. A semi-Markov process can be described in brief thus:

i) the individuals move from one state to another with random sojourn times in between;

ii) the succesive states visited form a Markov chain;

iii) the sojourn time has a distribution which depends on the state being visited as well as on the next state to be entered.

(For details, cf. Feller, 1964; Cinlar, 1975). Such a possibility has been explored during the last decade by analysts in various fields. The implications, both theoretical and practical, of working with a semi-Markov model in demography can be gainfully glimpsed through the three research papers presented by Ralph B.Ginsberg, Jan M.Hoem and Charles J.Mode.

The paper presented by Ginsberg (1971) suggests a model to capture the McGinnis' axiom of "cumulative inertia", though not restricted to it. According to this axiom, there is a strong and incrasing tendency for people to be retained in the state they occupy. Therefore, it would be more relevant to subject the probability of leaving a state to be dependent both on the length of time a state has been occupied and on the next state to be visited (the so-called "pulls" and "pushes", in contrast to the Markov process where only the push is considered). Ginsberg suggests the use of semi-Markov model and outlines the possibility of incorporating such factors as age, historical effects and other inhomogeneities.

When only duration in a state is considered, along with pulls and pushes, the semi-Markov model is said to be homogeneous or age-independent. A homogeneous model renders neat expressions for probability matrices; in particular, the Laplace transform makes easy the solution of these probability matrices. But when age, also an important factor in demographic analysis, is considered along with duration, computational complexity increases. Ginsberg suggets the device of operational time which transforms the inhomogeneous or age-dependent semi-Markov process into a homogeneous one.

Hoem (1972) presents a mathematical treatment of inhomogeneous semi-Markov processes from a sample path perspective and from a probabilistic point of view. He focuses his attention from the very start on the forces of transition and has recourse to the device of operational time suggested by Ginsberg. This approach leads to theoretically interesting results, but "tends to obscure what is being actually assumed, explicitly or implicitly, about sample paths".[1] Further, it is not clear how an algorithm could be developed for generating realizations of sample paths through the abstract probabilities given in his equations in Section 4.

Mode (1982) also treats the semi-Markov process from a sample path perspective but has recourse to the time-honoured but underutilized, theoretical advantages of the Kolmogorov backward differential equations (Feller, 1950, 1966). He suggests the possibility of extending the backward equations through the sample path perspective to include the case of sojourn time in states with arbitrary distributions.[2] This leads to the formation of renewal-type integral equations, in both age-dependent and age-independent cases. While the integral equations in the latter lead to an easy recursive solution, those in the former require an application of Littman's algorithm in their discrete time analogues (Littman and Mode, 1977).

The basic ideas underlying these three papers can be traced back, in one form or another, to earlier works of Feller (1950, 1964, 1966). The approach each paper takes, however, has advantages of its own; theoretical (in helping towards a clearer under-

1. Charles J. Mode (1982), p.540.
2. Backward equations have always been used for further mathematical manipulations in stochastic literature. Ginsberg (1971) also makes use of them in deriving the Laplace-Stieltjes transform of the transition probability matrices in the homogeneous case (p.245).

standing of concepts) and practical (in helping to develop a workable algorithm). From the practical point of view, the methodology suggested by Mode has been found to be the most helpful. As was explained briefly above, his methodology is built on the backward Kolmogorov equations which are based on consideration of the first move in a series of steps - a property which facilitates the introduction of sojourn time in states. Thus, the first passage probabilities (which are the probabilities of moving out of a state occupied for a certain length of time) are generated as preliminary steps to finding the state probabilities. In fact, these first passage probabilities seem to present a more relevant and more realistic picture than the state probabilities, and easily lend themselves to parametrization which can be used in the study of the effects of heterogeneity.

Finding the state probabilities via the first passage probabilities in the age-dependent semi-Markov model is done through the application of Littman algorithm. Without this algorithm, it would not be possible to build more realistic models incorporating age-dependent semi-Markov processes.

This paper tries to map out the implications of the methodology suggested by Mode, of the Littman algorithm without which an age-dependent semi-Markov model cannot possibly be applied, and of certain salient features not to be found in the usual Markov-generated life tables. All this is illustrated with the use of period data normally available to demographers. This complements the application of the same methodology and Littman algorithm to longitudinal data of the Taichung Medical IUD Experiment by Mode and Soyka (1980) and to longitudinal but truncated data of the work histories of the disabled by Hennessey (1980). The period data used here are of marital status in Belgium, 1970.

A brief review of the basic ideas on which the semi-Markov model is built in presented in Section 2. The application of the algorithm ensuing from these basic ideas to period data is illustrated in Section 3. Some salient features of this semi-Markov model are pointed out in Section 4. And the interesting results of an attempt at parametrizing the first passage probabilities are presented in Section 5. Possibilities of bringing a greater degree of heterogeneity into the semi-Markov model and further works envisaged are outlined in the last section.

2. A BRIEF REVIEW OF THE SEMI-MARKOV MODEL : MODE's FORMULATION

a) Kolmogorov equations extended to include sojourn times in states

The Kolmogorov differential equations are fundamental in any treatment of Markov chains. They are given as[3]:

$$\frac{\delta P_{ij}(s,t)}{\delta t} = -q_j(t).P_{ij}(s,t) + \sum_{k\neq j} P_{ik}(s,t).q_k(t).\ \Pi_{kj}(t) \qquad (1)$$

$$\frac{\delta P_{ij}(s,t)}{\delta s} = q_i(s).P_{ij}(s,t) - \sum_{k\neq i} q_i(s).\ \Pi_{ik}(s).P_{kj}(s,t) \qquad (2)$$

The first is called the forward differential equation , the second the backward differential equation . Both the forward and the backward equations are essentially equivalent. The forward equations are intuitively easier to understand, but require an additional assumption, though purely analytical in character, in their derivation. The backward equations are easier to deal with from a rigorous point of view because of the less restrictive assumptions used to establish their validity. (For details, cf. Feller, 1950,pp.470-78.)

When the forward and backward equations are expressed in a different form in order to introduce sojourn times in states, they become, in the case of the age-independent (homogeneous) case,

---

3. The q's and π's have their usual connotations, namely, q's are the intensity functions defined by $q_{ij}(s) = \underset{h\to 0}{Lt}\ P_{ij}(s,s+h)/h$ and $q_{ii} = \underset{h\to 0}{Lt}\ (1-P_{ii}(s,s+h))/h$, and $q_i = \sum_j q_{ij} = -q_{ii}$. And Π_{ij} is the conditional probability of going to $j\neq i$, given that the process leaves i.

$$p_{ij}(t) = \delta_{ij}.e^{-q_j t} + \sum_{k \neq j} \int_0^t p_{ik}(s).q_k.\ \Pi_{kj}.e^{-q_j(t-s)}\ ds \qquad (1a)$$

$$p_{ij}(t) = \delta_{ij}.e^{-q_i t} + \sum_{k \neq i} \int_0^t q_i.e^{-q_i s}\ .\ \Pi_{ik}.p_{kj}(t-s)\ ds \qquad (2a)$$

where $p_{ij}(t)$, the state probability, denotes the probability of being in state j within t time units given that the individual (or the process) was in state i at t=0. These two expressions of the Kolmogorov differential equations express the state probability as the sum of two complementary events in a better way than in their original form in (1) and (2). Their interpretations bring out the difference between the two equations.

First, consider the backward equation. Given that the process starts in state i at t=0, two complementary events are possible. (i) The process is still in state i at $t > 0$. In this case, j=i, and the probability of this event is $\exp(-q_i t)dt$. The kronecker delta (δ_{ij}) makes the probability zero when $j \neq i$. (ii) The process leaves the initial state i at least once during the interval $(0,t]$, $t > 0$. As $q_i.\exp(-q_i t)$ is the probability density function of exponential distribution, $q_i.\exp(-q_i s)ds$ denotes the probability of leaving the initial state i during a small time interval ds. Given that the process leaves i, Π_{ik} is the conditional probability that it moves to state $k \neq i$. Once the state k has been entered at time s, $p_{kj}(t-s)$ is the conditional probability of being in state j at time t. Integrating over s and summing over all $k \neq i$ yields the second term. The sum of these two complementary events constitutes the expression of the backward equation as given above.

On the other hand, in the expression of the forward equation, the two complementary events are as follows: (i) Given that the process starts in state i at t=0, the process is found in state k at time $s > 0$, which is denoted by $P_{ik}(s)$. Only the last move preceding time t is now taken into consideration. The probability of a move from state k has the density q_k, whatever be the sojourn time in state k at time s. Here, the memoryless property of the exponential distribution plays a crucial role.[4] Given that the process leaves state k, Π_{kj} is the conditional probability of a move to state j, and the probability of no further jump between s and t equals $\exp(-q_j(t-s))$. Integrating over s and summing over $k \neq j$ gives the second term. (ii) The second event of staying in the same state i is given by the first term, which has the same interpretation as in the backward equation.

In the evolution of techniques for constructing the Markov-generated increment-decrement life tables, it is the forward equation which has been made use of (Schoen & Land, 1979; Schoen, 1979; Krishnamoorthy, 1979; Keyfitz, 1980). This equation is based on considerations concerning the last move out of state k and on the memoryless property of the exponential distribution. Thus, if $\underset{\sim}{P}(x)$ is the state transition probability matrix, $\underset{\sim}{P}(x+t) = \underset{\sim}{P}(x).\exp(\underset{\sim}{Q}(x).t)$ for $t > 0$, provided an estimate of the matrix $\underset{\sim}{Q}(x)$ depending on age x is available. The use of the forward equation in constructing increment-decrement life tables

4. Explanation: If T_k is a random variable representing the sojourn time in state k, the distribution function of T_k is given by $P(T_k \leq t) = F_k(t) = 1-\exp(-q_k t)$, $t > 0$. Then the conditional probability that the process moves out of k during a small time interval (u,u+h), $h > 0$, given that it has been in k for u time units, $u > 0$, is given by $P(u < T_k \leq u+h \mid T_k > u) = \frac{F_k(u+h) - F_k(u)}{1 - F_k(u)}$ $= 1 - \exp(q_k h) \simeq q_k h$.

makes of them easy extensions of single decrement life tables and only involves substituting vectors for scalars. But it does not give any insight into the length of stay or sojourn times in different states.

The backward equation has always been held to be the "point of departure" in any further mathematical treatment associated with Markov chains. It is also the point of departure in the algorithm developed by Mode. His approach consists in defining the basic probabilities found in the expression of the backward equation directly on the framework of the idea of sample paths, and in constructing one-step transition probabilities through the application of the theory of competing risks.

b) One-step semi-Markov Transition Probabilities

From the sample path perspective, let X_n denote the state entered at the n-th step, Y_n the sojourn time in state X_{n-1} $(n \geq 1)$, and $A_{ij}(t)$ be the conditional probability of being in state j at time t given that the process was in state i at t=0, and stayed in state i for Y_n time units. Then,

$$P\left[X_n = j,\ Y_n \leq t\ \middle|\ X_{n-1} = i\right] = A_{ij}(t) \tag{3}$$

whereby $A_{ij}(t)$ is a one-step transition function. This is easily identified from the Markov Renewal Theory in the age-independent (homogeneous) case as equivalent to

$$A_{ij}(t) = \Pi_{ij}(1 - e^{-q_i t}) \tag{4}$$

where $\Pi_{ij} = q_{ij}/q_i$. From this, it follows that the distribution of sojourn time in state i is

$$A_i(t) = \sum_j A_{ij}(t) = 1 - e^{-q_i t} \tag{5}$$

And hence, $1-A_i(t)$ is the conditional probability that the process is still in i at time t given that it started in i at t=0. Let $a_{ij}(t)$ be the density of the transition function $A_{ij}(t)$; thus,

$$a_{ij}(t) = \frac{dA_{ij}(t)}{dt} = \Pi_{ij} \cdot q_i \cdot e^{-q_i t} \tag{6}$$

With these expressions coming from the sample path perspective, the backward equation can be expressed as

$$P_{ij}(t) = \delta_{ij}\left[1-A_i(t)\right] + \sum_{k \neq i} \int_0^t a_{ik}(s) \cdot P_{kj}(t-s)\,ds \tag{7a}$$

This formula requires only a minor modification when absorbing states are considered. Let the state space S be divided into S_1

of absorbing states and S_2 of transient states. When $i \in S_1$ of absorbing states, $A_{ii}(t) = 1$ and $A_{ij}(t) = 0$. When $i \in S_2$ and $j \in S_1$,

$$P_{ij}(t) = A_{ij}(t) + \sum_{k \neq i} \int_0^t a_{ik}(s).P_{kj}(t-s)ds \qquad (7b)$$

The equations (7) are called Renewal-type Integral Equations in the stochastic literature.

So far only the homogeneous case has been considered. This can be easily extended to the inhomogeneous (age-dependent) case, at least in theory.[5] In the inhomogeneous case, let the function $A_{ij}(x,t)$ denote the conditional probability that an individual aged x enters state i and makes a one-step transition to state j during the age interval $(x, x+t]$, $t > 0$. If i is an absorbing state, $A_{ii}(x,t) > 0$ and $A_{ij}(x,t) = 0$. If i is not an absorbing state, suppose that there are corresponding densities $a_{ij}(x,t)$. Extending the notations involved in equations (3) to (7), the integral equations become

$$P_{ij}(x,t) = \delta_{ij} \left[1 - A_i(x,t)\right] + \sum_{k \neq i} \int_0^t a_{ik}(x,s).P_{kj}(x+s,t-s)ds$$

$$\text{for } i,k,j \in S_2 \qquad (8a)$$

and

$$P_{ij}(x,t) = A_{ij}(x,t) + \sum_{k \neq i} \int_0^t a_{ik}(x,s).P_{kj}(x+s,t-s)ds$$

$$\text{for } i,k \in S_2 \text{ and } j \in S_1 \qquad (8b)$$

Though these integral equations have been easily extended to cover the case of age dependence, the computational complexity involved increases because of additional dimensionality now present and, in particular, because of the presence of later time points (x+s) in the second term on the right hand side.

5. For details, cf. Mode, 1982, pp.541-546.

(c) Application of the Theory of Competing Risks

Our attention is focussed here on the age-dependent case. According to the theory of competing risks, there are independent latent sojourn times T_{ij} with distribution functions $F_{ij}(t)$ governing not only what state is visited next but also the time when this visit occurs. Corresponding to this latent distribution function, there are also the density and risk functions given respectively by

$$f_{ij}(t) = \frac{dF_{ij}(t)}{dt} \quad \text{and} \quad \theta_{ij}(t) = \frac{f_{ij}(t)}{1-F_{ij}(t)} \quad .$$

Similarly in the age-dependent case, given that the state i is entered when the individual is aged x, the conditional latent distribution function associated with state $j \neq i$ is given by

$$F_{ij}(x,t) = \frac{F_{ij}(x+t) - F_{ij}(x)}{1 - F_{ij}(x)} \tag{9}$$

and its associated latent risk function is

$$\eta_{ij}(x,t) = \frac{f_{ij}(x,t)}{1 - F_{ij}(x,t)}$$

where $f_{ij}(x,t)$ is the partial derivative of $F_{ij}(x,t)$ with respect to t and hence is the density function. It can be shown from (9) that

$$1 - F_{ij}(x,t) = \frac{1 - F_{ij}(x+t)}{1 - F_{ij}(x)} \tag{10}$$

and hence $\quad \eta_{ij}(x,t) = \theta_{ij}(x+t) \quad$ (11)

This greatly simplifies the procedure directed at accomodating age-dependence in discrete time, as the conditional latent risk function η_{ij} is determined by merely translating the risk function η_{ij} as in (11). Substantively this means that the latent risk function of an individual, who entered state i when aged x, to move to state j before t time units is equivalent to the latent risk function of an individual aged x+t.

Defining a corresponding discretized risk function, say, $r_{ij}(x,t) = q_{ij}(x+t)$, we can show that

$$q_{ij}(x+t) = \frac{A_{ij}(x,t) - A_{ij}(x,t-1)}{1 - A_i(x,t-1)} \qquad (12)$$

Before developing the algorithm based on the relationship (12), four points need to be emphasized.

i) In terms of semi-Markov processes in discrete time, $q_{ij}(t)$ is the conditional probability of a move to state j by time t, given that the state i was entered at t=0 and the process was still in i at time (t-1). Similar interpretation holds good for the expression $q_{ij}(x+t)$ found in (12).

ii) How to obtain the estimates q_{ij} ? In the usual procedure for constructing the multistate life tables, the observed age-specific rates are made equal to the life table rates and to the intensities of transition. The same observed age-specific rates can be used to get the estimates of the conditional probabilities q_{ij} by utilizing actuarial methods for converting rates into probabilities. In demographic practice, the conversion of rates into probabilities is done mainly through the linearity or the exponential assumption. In the application that follows in this paper, the linearity assumption has been retained, so as to make comparisons possible with the results obtained from the application of Markov-generated life tables constructed with the same assumption.

iii) The transition probabilities A_{ij} are one-step transition probabilities. Therefore, caution should be exercised while fixing age intervals; if they are wide, say 5 years, then multiple steps among states may contaminate the data and the results. For this reason, q_{ij} above has been restricted to the age interval (x+t-1, x+t); otherwise, it can generally be defined over the interval $(x+t_{n-1}, x+t_n)$, $n \geq 1$. In the following application, the one year age interval has been retained.

iv) There is an obvious difficulty encountered when period data are used - age at entrance into a state is not usually known in such a case. However, multistate life tables can be constructed, in general, for each age x as if the process started in each different i at each age x. This procedure would make the final results of the state probabilities obtained through the semi-Markov process outlined here comparable to the results obatined through the "status-based" measures of the Markov process (Willekens et al., 1980). See Section 3 for comparative results.

Once the estimates q_{ij} have been obtained, they can be transformed into the estimates of the function A_{ij} through the following relationships:

$$\begin{aligned} \text{let} \quad q_i(x+t) &= \sum_j q_{ij}(x+t) \\ p_i(x+t) &= 1 - q_i(x+t) \\ w_i(x+t) &= p_i(x+1).p_i(x+2).......p_i(x+t), \\ &\qquad \text{letting } w_i(x) = 1. \\ \text{then, } A_{ij}(x,t) &= \sum_{k=1}^{t} w_i(x+k-1).q_{ij}(x+k), \text{ for } x \geq 0,\ t \geq 1. \end{aligned} \qquad (13)$$

It is worth noting that since no state is vacated immediately, $a_{ij}(x,0)=0$, and hence $A_{ij}(x,0)=0$. Also, in the discrete version,

$$\begin{aligned} a_{ij}(x,t) &= A_{ij}(x,t) - A_{ij}(x,t-1) \\ &= w_i(x+t-1).q_{ij}(x+t) \end{aligned} \qquad (14)$$

Further, expressing (8a) and (8b) in their discrete forms,

$$P_{ij}(x,t) = \delta_{ij}\left[1-A_i(x,t)\right] + \sum_{k\neq i}\sum_{s=0}^{t} a_{ik}(x,s).P_{kj}(x+s,t-s) \quad \ldots\ldots(15a)$$

$$P_{ij}(x,t) = A_{ij}(x,t) + \sum_{k\neq i}\sum_{s=0}^{t} a_{ik}(x,s).P_{kj}(x+s,t-s) \quad (15b)$$

Note that the right hand sides of the above equations do not allow a recursive calculation as they involve the later time points (x+s). It is this characteristic which differentiates the age-dependent semi-Markov model from the age-independent one and makes the former more complex in actual calculations. At this juncture, the algorithm developed by Littman (Littman & Mode, 1977; Mode & Pickens, 1979) comes quite handy to circumvent the difficulty.

To explain very briefly the Littman algorithm, consider an example. Suppose we were to calculate $P_{ij}(20,2)$. One can verify that this amounts to the expression $P_{ij}(20,2)=\sum_k a_{ik}(20,1).P_{kj}(21,1)$. Thus, to calculate $P_{ij}(20,2)$, one needs to know $P_{kj}(21,1)$, which denotes the probability that an individual who entered state k at age 21 will be found in state j one year later. Of all the individuals who enter state k at age 21, some would make a one-step transition to j and continue staying there; some others would make one-step transition to some state v and then make another one-step transition to j, all these within one year interval, etc. Thus, $P_{kj}(21,1)$ implies not only the one-step transitions but also multiple transitions. The densities associated with these multiple transitions are called renewal densities, as the process renews itself after the first one-step transition. These renewal densities are based on the one-step transition densities, and since the latter are known for all ages and for all durations, $P_{kj}(21,1)$ can be expressed in terms of these one-step transition densities or renewal densities. The Littman algorithm calculates the renewal densities through the one-step transition densities a_{ik}. And the algorithm is as follows:

$$m_{ij}(x,t) = a^{o}_{ij}(x,t) + \sum_{k} \sum_{s=0}^{t} m_{ik}(x,s).a_{kj}(x+s,t-s) \qquad (16)$$

for $k \in S_2$, where $a^{o}_{ij}(x,0)= \delta_{ij}$ and $a^{o}_{ij}(x,t)=0$ for $t \neq 0$. Note that the intermediate state k can only be of S_2 as no "renewal" takes place in the absorbing state. The system (16) is a recursive system in t for each x because $a_{ij}(x,0)=0$.

With these renewal densities, (15a) and (15b) can be reexpressed as

$$\begin{aligned} p_{ij}(x,t) &= \sum_k \sum_s m_{ik}(x,s).\delta_{kj} \left[1-A_k(x+s,t-s)\right] \\ &= \sum_s m_{ij}(x,s) \left[1-A_j(x+s,t-s)\right] \quad \text{for } i,k,j \in S_2 \qquad (17a) \end{aligned}$$

and,

$$p_{ij}(x,t) = \sum_k \sum_s m_{ik}(x,s).A_{kj}(x+s,t-s) \quad \text{for } i,k \in S_2 \text{ and } j \in S_1 \qquad (17b)$$

Before concluding this section, a final note on the semi-Markov process would be of some help in understanding the results obtained through its application in the following sections. in an age-independent semi-Markov process, the successive states visited (namely, the sequence $\{X_n\}$) form a Markov chain; and given this sequence, the successive sojourn times (namely, the sequence $\{Y_n\}$) are conditionally independent. On the other hand, in an age-dependent semi-Markov process, apart from the sequence $\{X_n\}$ which forms a Markov chain, the successive sequence of the state-age pairs of states visited and of the age of the individual at the n-th step (namely, the sequence $\{X_n, T_n\}$) also enjoys the Markov property; but the sequence $\{Y_n\}$ of sojourn times in states is neither independently distributed nor enjoys the Markov property. For details, cf. Cinlar (1975), ch.10 and Mode (1982) pp.543-46.

What has been said above about the transitions of a particular individual in a population is also true of a homogeneous population composed of individuals following the same stochastic process, or of a heterogeneous population in which different stochastic processes are followed.

3. APPLICATION TO BELGIAN CENSUS DATA, 1970

The census in question was conducted on the 31st, Dec., 1970 and provides population figures by each marital status. To obtain the count of transitions between marital states corresponding to this date, an average of the figures of transitions in the years 1970 and 1971 is taken. The transitions to widowhood are obtained from the number of deaths (of married persons) of the opposite sex, without having recourse to any correction for disparity in ages between the spouses. The present paper gives only the results of the analysis done with the data on females.

(a) Computer Problems

In the calculations involved, there are four matrices:

$\underset{\sim}{A}(x,t) = \left[A_{ij}(x,t)\right]$ - the matrix of one-step transition probabilities, also called first passage probabilities

$\underset{\sim}{a}(x,t) = \left[a_{ij}(x,t)\right]$ - the matrix of first passage densities

$\underset{\sim}{M}(x,t) = \left[m_{ij}(x,t)\right]$ - the matrix of renewal densities

$\underset{\sim}{P}(x,t) = \left[P_{ij}(x,t)\right]$ - the matrix of state probabilities

As A_{ij} are one-step transition probabilties, the use of one year age interval would be the best. Using the single year age intervals, from age 15 to age 70 which is open-ended, with 25 duration time-points, the four states of Never Married (NM), Presently Married (PM), Widowed (W) and Divorced (D) and the absorbing state Death (DH) would give matrices with arrays of (x, j, i, t)=(56,5,4,25). Obviously, the computer memory space required would be enormous, and some effort is required at reducing this call on memory space.

During the preliminary trials, 5-year age intervals were used and no obvious errors such as negative probabilities or probabilities greater than unity were encountered. Therefore, 5-year age groups can perhaps always be used, thus minimizing greatly the required memory space, provided care is taken that the probability requirements are not violated. A <u>via media</u> could also be tried, using a mixture of single and 5-year age intervals (e.g. using single years for ages between 20 and 30, and 5-years for the rest). The results thereof were also satisfactory.

When using the single year intervals, the following procedure was adopted. The computer program was divided into four parts:

Part 1 - calculates the observed rates from the data file, converts them into conditional probabilities q_{ij} through the linearity assumption and finds the stationary probabilities Π_{ij}. These results are stored in Tape1 and Tape2 respectively.

Part 2 - makes use of the q_{ij} from Tape1 to find the first passage probabilities A_{ij} and their densities a_{ij} and stores these results in Tape3 and Tape4 respectively. The arrays of the matrices $\underset{\sim}{A}$ and $\underset{\sim}{a}$ are kept to their full size, as these are required for calculating the $\underset{\sim}{M}$ and $\underset{\sim}{P}$ matrices.

Part 3 - makes use of the $\underset{\sim}{a}$ matrices from Tape4 to find the renewal densities, and these are stored in Tape5. The first array of the matrix $\underset{\sim}{M}$ is reduced to 36, that is, only up to age 50 inclusively, as ages beyond this limit are not of much interest in many domains of demographic analysis.

Part 4 - makes use of the A-values from Tape3 and m-values from Tape5 to find the final state probabilities. The first array of $\underset{\sim}{P}$ is also reduced to 36 as in the case of $\underset{\sim}{M}$.

Even after slpitting the whole job into four parts as above, the memory space required is still enormous. Thus, for example, the matrix $\underset{\sim}{A}$ with arrays (56,5,4,25) alone requires more than 200,000 CM, not normally available in a job with a CDC computer. Therefore, Parts 2 to 4 are made to work in two subdivisions with matrices of arrays half the size of what is necessary.

(b) An Illustrative Example

As an example from the computer output, Table 1 provides the first passage probabilities, Table 2 the renewal densities and Table 3 the state probabilities - for x, the age of entrance into the relevant states of interest, equal to 15 and 20.

Note that since certain direct transitions in our study are not possible, for example from the NM to D, the corresponding first passage probabilities are also zero. But the renewal densities are not zero, because once the direct transition is made to the PM from the NM, the process renews itself and passes from the PM to D within the same duration.

Since each age is taken as the age of entrance into state i, there will be a corresponding life table for each age x. In the Markov-generated multistate life table construction, a distinction is made between the population-based measures and the status-based measures. The status-based life table gives the expected number

Table 1.

```
********************************
* FIRST PASSAGE                *
* PROBABILITIES FOR EACH STATUS*
* ENTERED AT AGE X             *
* F I R S T   P A R T          *
********************************
```

AGE OF ENTRANCE INTO STATUS IS 15

AGE ***	NEV. MAR. ********					PRES. MAR ********					WIDOWED. ********					DIVORCED ********				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
15	0.000	.003	0.000	0.000	.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.000	.016	0.000	0.000	.001	0.000	0.000	0.000	0.000	.001	0.000	0.000	0.000	0.000	.105	0.000	0.000	0.000	0.000	0.000
17	0.000	.052	0.000	0.000	.001	0.000	0.000	0.000	.001	.001	0.000	0.000	0.000	0.000	.105	0.000	0.000	0.000	0.000	0.000
18	0.000	.130	0.000	0.000	.002	0.000	0.000	0.000	.001	.002	0.000	.028	0.000	0.000	.105	0.000	.095	0.000	0.000	0.000
19	0.000	.256	0.000	0.000	.002	0.000	0.000	.000	.003	.003	0.000	.078	0.000	0.000	.105	0.000	.126	0.000	0.000	0.000
20	0.000	.405	0.000	0.000	.003	0.000	0.000	.001	.004	.003	0.000	.156	0.000	0.000	.105	0.000	.307	0.000	0.000	0.000
21	0.000	.564	0.000	0.000	.003	0.000	0.000	.001	.007	.004	0.000	.233	0.000	0.000	.105	0.000	.418	0.000	0.000	0.000
22	0.000	.685	0.000	0.000	.003	0.000	0.000	.002	.010	.004	0.000	.299	0.000	0.000	.117	0.000	.565	0.000	0.000	.003
23	0.000	.775	0.000	0.000	.004	0.000	0.000	.003	.014	.004	0.000	.348	0.000	0.000	.125	0.000	.649	0.000	0.000	.006
24	0.000	.832	0.000	0.000	.004	0.000	0.000	.003	.019	.005	0.000	.412	0.000	0.000	.131	0.000	.710	0.000	0.000	.006
25	0.000	.871	0.000	0.000	.004	0.000	0.000	.004	.025	.005	0.000	.461	0.000	0.000	.133	0.000	.773	0.000	0.000	.006
26	0.000	.893	0.000	0.000	.004	0.000	0.000	.005	.031	.006	0.000	.498	0.000	0.000	.138	0.000	.818	0.000	0.000	.007
27	0.000	.908	0.000	0.000	.004	0.000	0.000	.006	.037	.006	0.000	.528	0.000	0.000	.139	0.000	.847	0.000	0.000	.007
28	0.000	.919	0.000	0.000	.005	0.000	0.000	.007	.043	.007	0.000	.551	0.000	0.000	.143	0.000	.872	0.000	0.000	.008
29	0.000	.928	0.000	0.000	.005	0.000	0.000	.008	.049	.007	0.000	.568	0.000	0.000	.145	0.000	.893	0.000	0.000	.008
30	0.000	.934	0.000	0.000	.005	0.000	0.000	.009	.054	.008	0.000	.587	0.000	0.000	.145	0.000	.906	0.000	0.000	.008
31	0.000	.938	0.000	0.000	.005	0.000	0.000	.010	.059	.008	0.000	.603	0.000	0.000	.145	0.000	.918	0.000	0.000	.008
32	0.000	.941	0.000	0.000	.005	0.000	0.000	.012	.064	.009	0.000	.614	0.000	0.000	.145	0.000	.928	0.000	0.000	.009
33	0.000	.944	0.000	0.000	.005	0.000	0.000	.013	.069	.010	0.000	.625	0.000	0.000	.147	0.000	.936	0.000	0.000	.009
34	0.000	.947	0.000	0.000	.005	0.000	0.000	.014	.073	.011	0.000	.637	0.000	0.000	.147	0.000	.943	0.000	0.000	.009
35	0.000	.949	0.000	0.000	.005	0.000	0.000	.015	.078	.012	0.000	.646	0.000	0.000	.148	0.000	.948	0.000	0.000	.009
36	0.000	.950	0.000	0.000	.006	0.000	0.000	.017	.081	.013	0.000	.653	0.000	0.000	.149	0.000	.953	0.000	0.000	.009
37	0.000	.952	0.000	0.000	.006	0.000	0.000	.018	.085	.014	0.000	.661	0.000	0.000	.150	0.000	.957	0.000	0.000	.009
38	0.000	.953	0.000	0.000	.006	0.000	0.000	.020	.088	.015	0.000	.667	0.000	0.000	.151	0.000	.960	0.000	0.000	.009
39	0.000	.954	0.000	0.000	.006	0.000	0.000	.021	.091	.016	0.000	.672	0.000	0.000	.151	0.000	.963	0.000	0.000	.010

AGE OF ENTRANCE INTO STATUS IS 20

AGE ***	NEV. MAR. ********					PRES. MAR ********					WIDOWED. ********					DIVORCED ********				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
20	0.000	.201	0.000	0.000	.001	0.000	0.000	.000	.002	.000	0.000	.095	0.000	0.000	0.000	0.000	.207	0.000	0.000	0.000
21	0.000	.414	0.000	0.000	.001	0.000	0.000	.001	.004	.001	0.000	.189	0.000	0.000	0.000	0.000	.334	0.000	0.000	0.000
22	0.000	.579	0.000	0.000	.002	0.000	0.000	.001	.007	.001	0.000	.271	0.000	0.000	.014	0.000	.502	0.000	0.000	.004
23	0.000	.700	0.000	0.000	.002	0.000	0.000	.002	.011	.002	0.000	.331	0.000	0.000	.024	0.000	.598	0.000	0.000	.007
24	0.000	.776	0.000	0.000	.002	0.000	0.000	.003	.016	.002	0.000	.409	0.000	0.000	.031	0.000	.668	0.000	0.000	.007
25	0.000	.829	0.000	0.000	.003	0.000	0.000	.004	.023	.003	0.000	.470	0.000	0.000	.034	0.000	.740	0.000	0.000	.007
26	0.000	.859	0.000	0.000	.003	0.000	0.000	.005	.029	.003	0.000	.514	0.000	0.000	.040	0.000	.792	0.000	0.000	.008
27	0.000	.879	0.000	0.000	.003	0.000	0.000	.006	.035	.004	0.000	.551	0.000	0.000	.041	0.000	.825	0.000	0.000	.009
28	0.000	.894	0.000	0.000	.003	0.000	0.000	.007	.041	.004	0.000	.579	0.000	0.000	.046	0.000	.854	0.000	0.000	.009
29	0.000	.905	0.000	0.000	.003	0.000	0.000	.008	.046	.005	0.000	.601	0.000	0.000	.049	0.000	.878	0.000	0.000	.009
30	0.000	.914	0.000	0.000	.004	0.000	0.000	.009	.051	.005	0.000	.624	0.000	0.000	.049	0.000	.893	0.000	0.000	.010
31	0.000	.919	0.000	0.000	.004	0.000	0.000	.010	.056	.006	0.000	.643	0.000	0.000	.049	0.000	.906	0.000	0.000	.010
32	0.000	.924	0.000	0.000	.004	0.000	0.000	.011	.061	.007	0.000	.656	0.000	0.000	.049	0.000	.918	0.000	0.000	.010
33	0.000	.928	0.000	0.000	.004	0.000	0.000	.012	.066	.007	0.000	.670	0.000	0.000	.051	0.000	.927	0.000	0.000	.010
34	0.000	.931	0.000	0.000	.004	0.000	0.000	.013	.071	.008	0.000	.684	0.000	0.000	.051	0.000	.935	0.000	0.000	.010
35	0.000	.933	0.000	0.000	.004	0.000	0.000	.015	.075	.009	0.000	.696	0.000	0.000	.053	0.000	.941	0.000	0.000	.010
36	0.000	.936	0.000	0.000	.005	0.000	0.000	.016	.079	.010	0.000	.705	0.000	0.000	.054	0.000	.946	0.000	0.000	.011
37	0.000	.938	0.000	0.000	.005	0.000	0.000	.018	.082	.011	0.000	.714	0.000	0.000	.055	0.000	.951	0.000	0.000	.011
38	0.000	.939	0.000	0.000	.005	0.000	0.000	.019	.086	.012	0.000	.722	0.000	0.000	.056	0.000	.954	0.000	0.000	.011
39	0.000	.940	0.000	0.000	.005	0.000	0.000	.021	.089	.013	0.000	.728	0.000	0.000	.057	0.000	.957	0.000	0.000	.011
40	0.000	.941	0.000	0.000	.005	0.000	0.000	.023	.092	.015	0.000	.734	0.000	0.000	.057	0.000	.960	0.000	0.000	.011
41	0.000	.942	0.000	0.000	.005	0.000	0.000	.026	.095	.016	0.000	.740	0.000	0.000	.058	0.000	.962	0.000	0.000	.011
42	0.000	.943	0.000	0.000	.006	0.000	0.000	.028	.097	.018	0.000	.745	0.000	0.000	.059	0.000	.964	0.000	0.000	.011
43	0.000	.944	0.000	0.000	.006	0.000	0.000	.031	.099	.019	0.000	.748	0.000	0.000	.060	0.000	.965	0.000	0.000	.012
44	0.000	.945	0.000	0.000	.006	0.000	0.000	.034	.101	.021	0.000	.752	0.000	0.000	.060	0.000	.967	0.000	0.000	.012

Table 2.

****
* RENEWAL DENSITIES *
* FOR EACH STATUS *
*ENTERED AT AGE X *
****

AGE OF ENTRANCE INTO STATUS IS 15

AGE	NEV. MAR.					PRES. MAR					WIDOWED.					DIVORCED				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
15	0.000	.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.000	.013	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
17	0.000	.036	0.000	.000	0.000	0.000	0.000	0.000	.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.000	.079	0.000	.000	0.000	0.000	.000	0.000	.001	0.000	0.000	.028	0.000	0.000	0.000	0.000	.095	0.000	0.000	0.000
19	0.000	.126	.000	.000	0.000	0.000	.000	.000	.001	0.000	0.000	.050	.000	.000	0.000	0.000	.031	.000	.000	0.000
20	0.000	.149	.000	.000	0.000	0.000	.001	.000	.002	0.000	0.000	.078	.000	.000	0.000	0.000	.181	.000	.000	0.000
21	0.000	.158	.000	.001	0.000	0.000	.001	.001	.002	0.000	0.000	.077	.000	.000	0.000	0.000	.111	.000	.001	0.000
22	0.000	.122	.000	.002	0.000	0.000	.002	.001	.003	0.000	0.000	.067	.000	.001	0.000	0.000	.147	.000	.001	0.000
23	0.000	.090	.000	.003	0.000	0.000	.001	.001	.004	0.000	0.000	.049	.000	.001	0.000	0.000	.084	.000	.002	0.000
24	0.000	.058	.001	.004	0.000	0.000	.002	.001	.005	0.000	0.000	.064	.000	.002	0.000	0.000	.062	.000	.003	0.000
25	0.000	.041	.001	.005	0.000	0.000	.003	.001	.006	0.000	0.000	.050	.000	.003	0.000	0.000	.064	.001	.005	0.000
26	0.000	.025	.001	.005	0.000	0.000	.004	.001	.006	0.000	0.000	.038	.000	.003	0.000	0.000	.047	.001	.005	0.000
27	0.000	.013	.001	.005	0.000	0.000	.004	.001	.006	0.000	0.000	.032	.001	.003	0.000	0.000	.031	.001	.005	0.000
28	0.000	.014	.001	.005	0.000	0.000	.004	.001	.006	0.000	0.000	.025	.001	.003	0.000	0.000	.028	.001	.005	0.000
29	0.000	.012	.001	.005	0.000	0.000	.005	.001	.006	0.000	0.000	.019	.001	.003	0.000	0.000	.025	.001	.005	0.000
30	0.000	.009	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.021	.001	.003	0.000	0.000	.016	.001	.005	0.000
31	0.000	.008	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.017	.001	.003	0.000	0.000	.015	.001	.005	0.000
32	0.000	.007	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.013	.001	.003	0.000	0.000	.014	.001	.005	0.000
33	0.000	.007	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.013	.001	.003	0.000	0.000	.012	.001	.005	0.000
34	0.000	.006	.001	.004	0.000	0.000	.004	.001	.005	0.000	0.000	.014	.001	.003	0.000	0.000	.010	.001	.004	0.000
35	0.000	.005	.001	.004	0.000	0.000	.004	.001	.005	0.000	0.000	.011	.001	.003	0.000	0.000	.009	.001	.004	0.000
36	0.000	.005	.001	.004	0.000	0.000	.004	.002	.004	0.000	0.000	.009	.001	.003	0.000	0.000	.008	.001	.004	0.000
37	0.000	.006	.002	.004	0.000	0.000	.004	.002	.004	0.000	0.000	.011	.001	.002	0.000	0.000	.008	.002	.004	0.000
38	0.000	.004	.002	.003	0.000	0.000	.003	.002	.004	0.000	0.000	.008	.001	.002	0.000	0.000	.006	.002	.004	0.000
39	0.000	.004	.002	.003	0.000	0.000	.003	.002	.004	0.000	0.000	.007	.001	.002	0.000	0.000	.006	.002	.003	0.000

AGE OF ENTRANCE INTO STATUS IS 20

AGE	NEV. MAR.					PRES. MAR					WIDOWED.					DIVORCED				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
20	0.000	.201	0.000	0.000	0.000	0.000	0.000	.000	.002	0.000	0.000	.095	0.000	0.000	0.000	0.000	.207	0.000	0.000	0.000
21	0.000	.213	.000	.000	0.000	0.000	.000	.001	.002	0.000	0.000	.094	.000	.000	0.000	0.000	.127	.000	.000	0.000
22	0.000	.164	.000	.001	0.000	0.000	.001	.001	.003	0.000	0.000	.082	.000	.001	0.000	0.000	.168	.000	.001	0.000
23	0.000	.121	.000	.002	0.000	0.000	.001	.001	.004	0.000	0.000	.060	.000	.001	0.000	0.000	.096	.000	.002	0.000
24	0.000	.077	.000	.004	0.000	0.000	.002	.001	.005	0.000	0.000	.078	.000	.002	0.000	0.000	.071	.000	.003	0.000
25	0.000	.054	.001	.005	0.000	0.000	.003	.001	.006	0.000	0.000	.061	.000	.003	0.000	0.000	.073	.001	.004	0.000
26	0.000	.032	.001	.005	0.000	0.000	.004	.001	.006	0.000	0.000	.046	.000	.003	0.000	0.000	.053	.001	.005	0.000
27	0.000	.023	.001	.005	0.000	0.000	.003	.001	.006	0.000	0.000	.038	.001	.003	0.000	0.000	.035	.001	.005	0.000
28	0.000	.018	.001	.005	0.000	0.000	.004	.001	.006	0.000	0.000	.030	.001	.003	0.000	0.000	.032	.001	.005	0.000
29	0.000	.015	.001	.005	0.000	0.000	.005	.001	.006	0.000	0.000	.023	.001	.003	0.000	0.000	.028	.001	.005	0.000
30	0.000	.011	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.025	.001	.003	0.000	0.000	.017	.001	.004	0.000
31	0.000	.009	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.021	.001	.003	0.000	0.000	.016	.001	.005	0.000
32	0.000	.008	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.016	.001	.003	0.000	0.000	.015	.001	.005	0.000
33	0.000	.007	.001	.005	0.000	0.000	.004	.001	.005	0.000	0.000	.015	.001	.003	0.000	0.000	.013	.001	.005	0.000
34	0.000	.007	.001	.004	0.000	0.000	.004	.001	.005	0.000	0.000	.017	.001	.003	0.000	0.000	.011	.001	.004	0.000
35	0.000	.006	.001	.004	0.000	0.000	.004	.001	.005	0.000	0.000	.013	.001	.003	0.000	0.000	.009	.001	.004	0.000
36	0.000	.006	.001	.004	0.000	0.000	.004	.002	.004	0.000	0.000	.011	.001	.003	0.000	0.000	.008	.001	.004	0.000
37	0.000	.006	.002	.003	0.000	0.000	.004	.002	.004	0.000	0.000	.012	.001	.003	0.000	0.000	.009	.002	.003	0.000
38	0.000	.004	.002	.003	0.000	0.000	.003	.002	.004	0.000	0.000	.009	.001	.003	0.000	0.000	.006	.002	.003	0.000
39	0.000	.004	.002	.003	0.000	0.000	.003	.002	.004	0.000	0.000	.008	.001	.003	0.000	0.000	.006	.002	.003	0.000
40	0.000	.004	.002	.003	0.000	0.000	.003	.002	.003	0.000	0.000	.008	.002	.002	0.000	0.000	.005	.002	.003	0.000
41	0.000	.004	.002	.003	0.000	0.000	.003	.002	.003	0.000	0.000	.009	.002	.002	0.000	0.000	.005	.002	.003	0.000
42	0.000	.004	.003	.003	0.000	0.000	.003	.003	.003	0.000	0.000	.006	.002	.002	0.000	0.000	.005	.003	.003	0.000
43	0.000	.003	.003	.002	0.000	0.000	.002	.003	.002	0.000	0.000	.005	.002	.002	0.000	0.000	.004	.003	.002	0.000
44	0.000	.003	.003	.002	0.000	0.000	.003	.004	.002	0.000	0.000	.006	.003	.002	0.000	0.000	.004	.003	.002	0.000

Table 3.

****

* STATE PROBABILITIES FOR EACH STATUS*

* ENTERED AT AGE X *

****

AGE OF ENTRANCE INTO STATUS IS 15

AGE	NEV. MAR.					PRES. MAR					WIDOWED.					DIVORCED				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
15	.997	.003	0.000	0.000	.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000
16	.984	.015	0.000	0.000	.001	0.000	.999	0.000	0.000	.001	0.000	0.000	.895	0.000	.105	0.000	0.000	0.000	1.000	0.000
17	.947	.052	0.000	.000	.001	0.000	.998	0.000	.001	.001	0.000	0.000	.895	0.000	.105	0.000	0.000	0.000	1.000	0.000
18	.868	.130	0.000	.000	.002	0.000	.996	0.000	.001	.002	0.000	.028	.866	0.000	.105	0.000	.095	0.000	.905	0.000
19	.741	.256	.000	.000	.002	0.000	.994	.000	.003	.003	0.000	.078	.817	.000	.105	0.000	.126	.000	.874	.000
20	.592	.404	.000	.001	.003	0.000	.992	.001	.004	.003	0.000	.155	.739	.000	.105	0.000	.306	.000	.694	.000
21	.433	.561	.000	.001	.003	0.000	.990	.001	.006	.004	0.000	.232	.662	.001	.105	0.000	.416	.000	.583	.000
22	.311	.681	.001	.003	.004	0.000	.987	.002	.007	.004	0.000	.298	.584	.001	.117	0.000	.562	.000	.434	.004
23	.221	.768	.001	.005	.005	0.000	.984	.002	.010	.005	0.000	.346	.527	.002	.125	0.000	.643	.001	.349	.007
24	.164	.821	.001	.008	.005	0.000	.980	.003	.013	.005	0.000	.407	.458	.003	.131	0.000	.702	.001	.290	.007
25	.125	.856	.002	.012	.006	0.000	.975	.003	.017	.005	0.000	.454	.407	.005	.134	0.000	.760	.002	.230	.008
26	.103	.874	.003	.015	.006	0.000	.971	.004	.019	.006	0.000	.488	.366	.007	.139	0.000	.801	.002	.188	.008
27	.087	.885	.003	.018	.007	0.000	.967	.005	.022	.007	0.000	.516	.335	.009	.140	0.000	.826	.003	.161	.010
28	.076	.892	.004	.020	.008	0.000	.963	.005	.024	.007	0.000	.537	.308	.011	.144	0.000	.848	.004	.138	.011
29	.068	.897	.005	.022	.009	0.000	.961	.006	.026	.008	0.000	.552	.289	.012	.147	0.000	.866	.004	.118	.011
30	.061	.900	.005	.023	.009	0.000	.958	.007	.027	.009	0.000	.569	.271	.013	.147	0.000	.876	.005	.107	.012
31	.057	.902	.006	.025	.010	0.000	.955	.007	.029	.009	0.000	.582	.256	.015	.148	0.000	.884	.006	.097	.013
32	.053	.902	.007	.027	.011	0.000	.952	.008	.030	.010	0.000	.591	.245	.016	.149	0.000	.891	.007	.088	.014
33	.050	.902	.008	.028	.012	0.000	.949	.009	.031	.011	0.000	.600	.233	.017	.151	0.000	.896	.007	.081	.015
34	.048	.901	.008	.029	.013	0.000	.947	.010	.032	.012	0.000	.610	.221	.018	.152	0.000	.900	.008	.076	.016
35	.046	.900	.009	.030	.014	0.000	.944	.011	.033	.013	0.000	.617	.211	.019	.153	0.000	.902	.009	.072	.017
36	.044	.899	.011	.030	.015	0.000	.941	.012	.033	.014	0.000	.622	.204	.019	.155	0.000	.904	.010	.068	.018
37	.043	.899	.012	.030	.017	0.000	.939	.013	.033	.015	0.000	.628	.196	.019	.157	0.000	.906	.011	.063	.020
38	.042	.897	.013	.031	.018	0.000	.936	.014	.033	.017	0.000	.632	.190	.020	.158	0.000	.906	.012	.061	.021
39	.041	.894	.014	.032	.020	0.000	.932	.016	.034	.018	0.000	.634	.185	.020	.160	0.000	.905	.014	.059	.023

AGE OF ENTRANCE INTO STATUS IS 20

AGE	NEV. MAR.					PRES. MAR					WIDOWED.					DIVORCED				
X+T	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH	NM	PM	W	D	DH
20	.798	.201	0.000	0.000	.001	0.000	.998	.000	.002	.000	0.000	.095	.905	0.000	0.000	0.000	.207	0.000	.793	0.000
21	.585	.414	.000	.000	.001	0.000	.994	.001	.004	.001	0.000	.189	.811	.000	.000	0.000	.333	.000	.667	.000
22	.420	.576	.000	.002	.002	0.000	.992	.001	.006	.001	0.000	.270	.715	.001	.014	0.000	.500	.000	.495	.004
23	.293	.695	.001	.004	.003	0.000	.988	.002	.009	.002	0.000	.329	.645	.002	.024	0.000	.594	.001	.398	.007
24	.221	.768	.001	.007	.003	0.000	.983	.002	.012	.002	0.000	.405	.560	.003	.032	0.000	.661	.001	.330	.007
25	.168	.816	.002	.010	.004	0.000	.978	.003	.016	.003	0.000	.463	.497	.005	.034	0.000	.729	.001	.261	.008
26	.138	.841	.002	.013	.005	0.000	.974	.004	.019	.003	0.000	.505	.448	.007	.040	0.000	.777	.002	.212	.009
27	.118	.858	.003	.016	.005	0.000	.970	.004	.022	.004	0.000	.539	.409	.009	.042	0.000	.806	.003	.181	.010
28	.103	.869	.004	.019	.006	0.000	.966	.005	.024	.005	0.000	.565	.377	.011	.047	0.000	.831	.003	.154	.011
29	.091	.877	.005	.020	.007	0.000	.964	.006	.025	.005	0.000	.584	.353	.012	.050	0.000	.852	.004	.131	.012
30	.083	.882	.005	.022	.008	0.000	.961	.006	.027	.006	0.000	.605	.331	.014	.051	0.000	.864	.005	.118	.013
31	.077	.885	.006	.024	.008	0.000	.958	.007	.029	.006	0.000	.621	.312	.015	.051	0.000	.874	.005	.107	.014
32	.072	.886	.007	.025	.009	0.000	.955	.008	.030	.007	0.000	.633	.299	.016	.052	0.000	.882	.006	.096	.015
33	.068	.887	.007	.027	.010	0.000	.952	.009	.031	.008	0.000	.643	.284	.018	.055	0.000	.888	.007	.089	.016
34	.065	.888	.008	.028	.011	0.000	.949	.010	.032	.009	0.000	.656	.270	.019	.056	0.000	.893	.008	.082	.017
35	.062	.887	.009	.029	.012	0.000	.946	.011	.033	.010	0.000	.665	.258	.020	.058	0.000	.896	.009	.077	.018
36	.060	.887	.010	.030	.014	0.000	.944	.012	.033	.011	0.000	.672	.248	.020	.060	0.000	.898	.010	.073	.019
37	.058	.887	.011	.029	.015	0.000	.942	.013	.033	.013	0.000	.679	.239	.020	.062	0.000	.901	.011	.067	.021
38	.056	.885	.012	.030	.016	0.000	.939	.014	.034	.014	0.000	.684	.231	.021	.063	0.000	.901	.012	.065	.022
39	.055	.883	.014	.031	.018	0.000	.935	.016	.034	.016	0.000	.687	.225	.022	.065	0.000	.901	.014	.062	.024
40	.053	.880	.016	.031	.020	0.000	.931	.017	.034	.017	0.000	.690	.220	.022	.067	0.000	.899	.015	.060	.025
41	.052	.877	.017	.032	.021	0.000	.927	.019	.035	.019	0.000	.694	.214	.023	.069	0.000	.897	.017	.058	.027
42	.051	.874	.019	.032	.024	0.000	.923	.021	.035	.021	0.000	.695	.211	.023	.071	0.000	.895	.019	.056	.029
43	.050	.871	.022	.032	.026	0.000	.917	.024	.035	.023	0.000	.695	.208	.023	.074	0.000	.892	.022	.055	.031
44	.049	.866	.025	.032	.028	0.000	.913	.027	.035	.025	0.000	.695	.206	.023	.076	0.000	.888	.025	.054	.033

Table 4. Expected Number of Survivors - Markov & Semi-Markov Models

AGE ***   INITIAL STATUS OF COHORT NEV. MAR.

AGE	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	1000.	0.	0.	0.
21	999.	798.	201.	0.	0.
22	999.	585.	413.	0.	1.
23	998.	420.	576.	0.	2.
24	997.	299.	694.	1.	4.
25	997.	222.	767.	1.	7.
26	996.	169.	816.	2.	10.
27	996.	139.	842.	2.	13.
28	995.	118.	859.	3.	15.
29	994.	103.	870.	4.	17.
30	993.	91.	879.	4.	19.
31	993.	83.	884.	5.	21.
32	992.	77.	887.	6.	23.
33	991.	72.	888.	7.	24.
34	990.	68.	889.	7.	25.
35	989.	65.	890.	8.	26.
36	988.	62.	889.	9.	27.
37	987.	60.	889.	10.	28.
38	985.	58.	889.	11.	28.
39	984.	56.	887.	12.	29.
40	982.	55.	885.	14.	29.
41	981.	54.	882.	15.	29.
42	979.	53.	879.	17.	30.
43	977.	51.	876.	19.	30.
44	975.	50.	872.	21.	30.
45	972.	50.	868.	24.	30.

AGE X+T ***   AGE OF ENRTY INTO NEV. MAR. IS 20

AGE X+T	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	1000.	0.	0.	0.
21	999.	798.	201.	0.	0.
22	999.	585.	414.	0.	0.
23	998.	420.	576.	0.	2.
24	997.	298.	695.	1.	4.
25	997.	221.	768.	1.	7.
26	996.	168.	816.	2.	10.
27	995.	138.	841.	2.	13.
28	995.	118.	858.	3.	16.
29	994.	103.	869.	4.	19.
30	993.	91.	877.	5.	20.
31	992.	83.	882.	5.	22.
32	992.	77.	885.	6.	24.
33	991.	72.	886.	7.	25.
34	990.	68.	887.	7.	27.
35	989.	65.	888.	8.	28.
36	988.	62.	887.	9.	29.
37	986.	60.	887.	10.	30.
38	985.	58.	887.	11.	29.
39	984.	56.	885.	12.	30.
40	982.	55.	883.	14.	31.
41	980.	53.	880.	16.	31.
42	979.	52.	877.	17.	32.
43	976.	51.	874.	19.	32.
44	974.	50.	871.	22.	32.
45	972.	49.	866.	25.	32.

AGE ***   INITIAL STATUS OF COHORT PRES. MAR

AGE	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	1000.	0.	0.
21	1000.	0.	998.	0.	2.
22	999.	0.	995.	1.	3.
23	999.	0.	992.	1.	5.
24	998.	0	989.	2.	8.
25	998.	0.	985	2.	11.
26	997.	0	980.	3.	14
27	997	0.	976.	3.	17.
28	996	0.	972	4.	20.
29	995.	0.	969	5.	22.
30	995.	0.	966.	6.	23.
31	994.	0.	963.	6.	25.
32	994.	0.	960.	7	26.
33	993	0	957	8.	27.
34	992	0	954.	9.	29.
35	991	0.	952	9.	30.
36	990	0	949	10.	31
37	989.	0	946.	11.	31
38	987.	0	944.	12.	31.
39	986.	0	941	14.	31.
40	984.	0	937	15.	32.
41	983.	0	934	17.	32.
42	981.	0	930	19.	33.
43	979	0.	925	21.	33.
44	977.	0.	921	23.	33.
45	975	0.	916	26.	33.

AGE X+T ***   AGE OF ENRTY INTO PRES. MAR IS 20

AGE X+T	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	1000.	0.	0.
21	1000.	0.	998.	0.	2.
22	999.	0.	994.	1.	4.
23	999.	0.	992.	1.	6.
24	998.	0.	988.	2.	9.
25	998.	0.	983.	2.	12.
26	997.	0.	978.	3.	16.
27	997.	0.	974.	4.	19.
28	996.	0.	970.	4.	22.
29	995.	0.	966.	5.	24.
30	995.	0.	964.	6.	25.
31	994.	0.	961.	6.	27.
32	994.	0.	958.	7.	29.
33	993.	0.	955.	8.	30.
34	992.	0.	952.	9.	31.
35	991.	0.	949.	10.	32.
36	990.	0.	946.	11.	33.
37	989.	0.	944.	12.	33.
38	987.	0.	942.	13.	33.
39	986.	0.	939.	14.	34.
40	984.	0.	935.	16.	34.
41	983.	0.	931.	17.	34.
42	981.	0.	927.	19.	35.
43	979.	0.	923.	21.	35.
44	977.	0.	919.	24.	35.
45	975.	0.	913.	27.	35.

Table 4. Expected Number of Survivors -Markov and semi-Markov models

INITIAL STATUS OF COHORT WIDOWED.

AGE	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	0.	1000.	0.
21	1000.	0.	95.	905.	0.
22	1000.	0.	189.	811.	0.
23	986.	0.	269.	716.	1.
24	977.	0.	328.	647.	2.
25	970.	0.	403.	563.	3.
26	967.	0.	462.	500.	5.
27	961.	0.	504.	451.	7.
28	959.	0.	539.	412.	9.
29	955.	0.	565.	380.	10.
30	952.	0.	584.	356.	11.
31	951.	0.	605.	333.	13.
32	951.	0.	622.	315.	14.
33	950.	0.	633.	301.	15.
34	947.	0.	644.	287.	17.
35	946.	0.	657.	272.	18.
36	944.	0.	666.	260.	19.
37	942.	0.	672.	250.	19.
38	940.	0.	680.	241.	19.
39	939.	0.	685.	233.	20.
40	937.	0.	689.	227.	21.
41	935.	0.	691.	222.	21.
42	933.	0.	695.	216.	22.
43	931.	0.	696.	212.	22.
44	928.	0.	696.	210.	22.
45	926.	0.	696.	207.	23.

INITIAL STATUS OF COHORT DIVORCED

AGE	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	0.	0.	1000.
21	1000.	0.	207.	0.	793.
22	1000.	0.	333.	0.	667.
23	996.	0.	499.	0.	496.
24	993.	0.	593.	1.	400.
25	993.	0.	661.	1.	331.
26	992.	0.	729.	2.	262.
27	991.	0.	777.	2.	212.
28	990.	0.	806.	3.	181.
29	989.	0.	832.	3.	154.
30	989.	0.	854.	4.	131.
31	988.	0.	865.	5.	118.
32	987.	0.	876.	5.	106.
33	986.	0.	884.	6.	96.
34	985.	0.	890.	7.	88.
35	984.	0.	895.	8.	81.
36	983.	0.	898.	9.	76.
37	982.	0.	901.	10.	71.
38	980.	0.	904.	11.	66.
39	979.	0.	904.	12.	63.
40	977.	0.	903.	13.	61.
41	976.	0.	902.	15.	59.
42	974.	0.	900.	17.	57.
43	972.	0.	897.	19.	55.
44	970.	0.	894.	22.	54.
45	967.	0.	891.	24.	52.

AGE OF ENRTY INTO WIDOWED. IS 20

AGE X+T	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	0.	1000.	0.
21	1000.	0.	95.	905.	0.
22	1000.	0.	189.	811.	0.
23	986.	0.	270.	715.	1.
24	976.	0.	329.	645.	2.
25	968.	0.	405.	560.	3.
26	966.	0.	463.	497.	5.
27	960.	0.	505.	448.	7.
28	958.	0.	539.	409.	9.
29	953.	0.	565.	377.	11.
30	950.	0.	584.	353.	12.
31	949.	0.	605.	331.	14.
32	949.	0.	621.	312.	15.
33	948.	0.	633.	299.	16.
34	945.	0.	643.	284.	18.
35	944.	0.	656.	270.	19.
36	942.	0.	665.	258.	20.
37	940.	0.	672.	248.	20.
38	938.	0.	679.	239.	20.
39	937.	0.	684.	231.	21.
40	935.	0.	687.	225.	22.
41	933.	0.	690.	220.	22.
42	931.	0.	694.	214.	23.
43	929.	0.	695.	211.	23.
44	926.	0.	695.	208.	23.
45	924.	0.	695.	206.	23.

AGE OF ENRTY INTO DIVORCED IS 20

AGE X+T	TOTAL	NEV. MAR.	PRES. MAR	WIDOWED.	DIVORCED
20	1000.	0.	0.	0.	1000.
21	1000.	0.	207.	0.	793.
22	1000.	0.	333.	0.	667.
23	996.	0.	500.	0.	495.
24	993.	0.	594.	1.	398.
25	993.	0.	661.	1.	330.
26	992.	0.	729.	1.	261.
27	991.	0.	777.	2.	212.
28	990.	0.	806.	3.	181.
29	989.	0.	831.	3.	154.
30	988.	0.	852.	4.	131.
31	987.	0.	864.	5.	118.
32	986.	0.	874.	5.	107.
33	985.	0.	882.	6.	96.
34	984.	0.	888.	7.	89.
35	983.	0.	893.	8.	82.
36	982.	0.	896.	9.	77.
37	981.	0.	898.	10.	73.
38	979.	0.	901.	11.	67.
39	978.	0.	901.	12.	65.
40	976.	0.	901.	14.	62.
41	975.	0.	899.	15.	60.
42	973.	0.	897.	17.	58.
43	971.	0.	895.	19.	56.
44	969.	0.	892.	22.	55.
45	967.	0.	888.	25.	54.

of survivors at each age x', for those who are found in a particular status at a specified starting age x (x' > x). As the sequence of states visited in a semi-Markov process forms a Markov chain, the final results of state probabilities obtained through the application of the semi-Markov process outlined here will be the same as the expected number of survivors in the life table obtained through the status-based approach of the Markov process for the same starting age x. The results can be compared for the age of entrance x=20 in Table 4. The two tables correspond very closely because of single year intervals; if 5-year intervals or mixed intervals were to be used, one can expect some differences between the two.

While the states PM, W and D can be entered at any age x, the state NM admits in reality only one age of entrance x, say, 0 or 15. Hence, there is a sort of ambiguity in talking about age of entrance into the NM as equal to, say, 30 or 40. However, this notion is still of some use, as the probability of a NM person moving to the PM state increases up to a certain age if he is still not married by then. For this reason and also for reasons of uniformity in structure, each age x is considered also for the NM.

As an illustration of how the calculations are carried out, consider the age of entrance into state i at x=20. The four marital states are denoted by: NM = 1, PM = 2, W = 3, and D = 4; the absorbing state DH = 5. The following table gives the preliminary steps involved in the procedure given on page 15.

Age	transitions from i	to j	observed age-spec. rates R_{ij}	cond.prob. q_{ij}	$q_i = q_{ij}$	$p_i = 1 - q_i$	w_i
20	1	2	.223422	.200971	.201662	.798338	.798338
	1	5	.000691	.000691			
$t \leq 1$	2	3	.000340	.000340	.002419	.997581	.997581
	2	4	.001698	.001697			
	2	5	.000382	.000382			
	3	2	.100000	.095238	.095238	.904762	.904762
	3	5	.000000	.000000			
	4	2	.230769	.206896	.206896	.793104	.793104
21	1	2	.308374	.267137	.267788	.732212	.584553
	1	5	.000651	.000651			
$t \leq 2$	2	3	.000530	.000530	.003560	.996440	.994030
	2	4	.002415	.002412			
	2	5	.000618	.000618			
	3	2	.109756	.104046	.104046	.895954	.810625
	3	5	.000000	.000000			
	4	2	.173913	.160000	.160000	.840000	.666207
	4	5	.000000	.000000			
22	1	2	.326947	.281009	.281943	.718057	.419742
	1	5	.000934	.000934			
$t \leq 3$	2	3	.000606	.000606	.003880	.996120	.990173
	2	4	.002919	.002915			
	2	5	.000359	.000359			
	3	2	.106195	.100841	.118385	.881615	.714659
	3	5	.017699	.017544			
	4	2	.289157	.252632	.258638	.741362	.493901
	4	5	.006024	.006006			
23	1	2	.337241	.288580	.289522	.710478	.298218
	1	5	.000942	.000942			
$t \leq 4$	2	3	.000619	.000619	.005216	.994784	.985008
	2	4	.004200	.004191			
	2	5	.000406	.000406			
	3	2	.087838	.084143	.097566	.902434	.644933
	3	5	.013514	.013423			
	4	2	.215297	.194373	.200023	.799977	.395109
	4	5	.005666	.005650			

Note: The conditional probabilities q_{ij} have been calculated by the linearity assumption by which

$$q_{ij} = (2 * R_{ij}) / (2 + R_{ij})$$

R's and q's are rates and probabilities of transition between ages (x,x+1). Therefore, when we consider the age of entry to be x=20, this has the implication of duration t 1, t 2 etc. for successive ages.

Also, as w's are successive products of p's, we have, e.g. $w_1(21) = .798338 * .732212$, $w_1(22) = w(21) * .718057$, etc.

Once these preliminary calculations have been done, the first passage probabilities can be found out as follows:

$$A_{ij}(x,t) = \sum_{k=1}^{t} w_i(x+k-2).q_{ij}(x+k-1) \quad \text{letting } w_i(x-1)=1.$$

thus,

$$A_{ij}(20,t) = w_i(20+k-2).q_{ij}(20+k-1)$$

$$A_{ij}(20,1) = w_i(19).q_{ij}(20) = q_{ij}(20)$$

$$A_{ij}(20,2) = w_i(19).q_{ij}(20) + w_i(20).q_{ij}(21) = A_{ij}(20,1) + w_i(20).q_{ij}(21)$$

$$A_{ij}(20,3) = A_{ij}(20,2) + w_i(21).q_{ij}(22)$$

....etc. The following table presents the first passage probabilities and densities.

First Passage probabilities(A's) and their densities (a's), and the renewal densities (m's) for age of entrance into x=20

Transition from i	to j	t	$A_{ij}(20,t)$	$a_{ij}(20,t)$	$m_{ij}(20,t)$
1	2	1	.200971	.200971	.200971
1	3		.000000	.000000	.000000
1	4		.000000	.000000	.000000
1	5		.000691	.000691	-
2	3		.000340	.000340	.000340
2	4		.001697	.001697	.001697
2	5		.000382	.000382	-
3	2		.095238	.095238	.095238
3	4		.000000	.000000	.000000
3	5		.000000	.000000	-
4	2		.206896	.206896	.206896
4	3		.000000	.000000	.000000
4	5		.000000	.000000	-
1	2	2	.414237	.213266	.213266
1	3		.000000	.000000	.000106
1	4		.000000	.000000	.000484
1	5		.001211	.000520	-
2	3		.000869	.000529	.000529
2	4		.004103	.002406	.002406
2	5		.000999	.000617	-
3	2		.189375	.094137	.094137
3	4		.000000	.000000	.000130
3	5		.000000	.000000	-
4	2		.333793	.126897	.126897
4	3		.000000	.000000	.000109
4	5		.000000	.000000	-

and so on. Note that $a_{ij}(x,t) = A_{ij}(x,t) - A_{ij}(x,t-1)$ and hence, for example, $a_{32}(20,2) = A_{32}(20,2) - A_{32}(20,1) = .189375 - .095238 = .094137$.

Note also that where the first passage probabilities are zero, the renewal densities are not zero. Thus, for example,

$$m_{13}(20,2) = m_{11}(20,0).a_{13}(20,2) + m_{11}(20,1).a_{13}(21,1) + m_{12}(20,1).m_{23}(21,1) + m_{13}(20,1).a_{33}(21,1) + m_{14}(20,1).a_{43}(21,1)$$

$$= m_{11}(20,0).a_{13}(20,2) + m_{12}(20,1).a_{23}(21,1)$$

$$= 0 + .200972 * .000530 = .000106$$

Two points are worth noting in calculating the first passage densities (or probabilities) and the renewal densities.

i) Suppose we were to calculate $A_{ij}(21,t)$. Applying the same procedure, first we have $q_i(21+t-1)$, then $p_i(21+t-1)$. From this, we find $w_i(21+t-1)=p_i(21).p_i(22)......p_i(21+t-1)$. Thus, e.g. $w_i(22) = p_i(21).p_i(22)$. This value of $w_i(22)$ is <u>not the same</u> as $w_i(22)$ calculated for the age of entrance x=20; here the age of entrance is x=21. Thus, $w_2(22)$ for x=20 is .990173 while $w_2(22)$ for x=21 is .992574. The difference lies in the fact that $w_i(22)$ for x=20 is given by $p_i(20).p_i(21).p_i(22)$.

ii) In calculating the renewal densities, the summation over s ranges from 0 to t. When s=t, $a_{kj}(x+s,t-s) = a_{kj}(x+s,0) = 0$. Therefore, we can completely neglect the last term. Further, for all x and t, $a_{ii}=0$. Thus, the formula specified in (12) can be simplified to

$$m_{ij}(x,t) = m_{ii}(x,0).a_{ij}(x,t) + \sum_{k\neq j} \sum_{s=1}^{t-1} m_{ik}(x,s).a_{kj}(x+s,t-s) \quad (14)$$

For example, to go beyond the specifications of the table,

$$\begin{aligned} m_{32}(20,3) &= m_{33}(20,0).a_{32}(20,3) + m_{31}(20,1).a_{12}(21,2) + \\ &\quad m_{33}(20,1).a_{32}(21,2) + m_{34}(20,1).a_{42}(21,2) + \\ &\quad m_{31}(20,2).a_{12}(22,1) + m_{33}(20,2).a_{32}(22,1) + \\ &\quad m_{34}(20,2).a_{42}(22,1) \\ &= .081744 + (0. * .205759) + (0. * .090348) + \\ &\quad + (0. * ..) + (0. * ...) + (.000051 * .100084) \\ &\quad + (.000229 * .252632) \\ &= .081807 \end{aligned}$$

Note also that renewal densities do not exist when j =5, namely death, the absorbing state.

Once the values of A and m have been obtained, the state probabilities P can be calculated. Again, when s=t, $A_j(x+s,t-s)$ do not exist, and the formula (13a) and (13b) can be simplified to:

$$p_{ij}(x,t) = \sum_{s=0}^{t} m_{ij}(x,s) \left[1-A_j(x+s,t-s)\right] \qquad \text{for } i,j \in S_2$$

$$= \sum_{s=0}^{t-1} m_{ij}(x,s) \left[1-A_j(x+s,t-s)\right] + m_{ij}(x,t)$$

$$= m_{ij}(x,0) \left[1-A_j(x,t)\right] + \sum_{s=1}^{t-1} m_{ij}(x,s) \left[1-A_j(x+s,t-s)\right] + m_{ij}(x,t) \qquad (15a)$$

where $m_{ij}(x,0) = 1$, for $i=j$

$m_{ij}(x,0) = 0$, for $i \neq j$

Similarly,

$$p_{ij}(x,t) = A_{ij}(x,t) + \sum_k \sum_{s=1}^{t-1} m_{ik}(x,s) . A_{kj}(x+s,t-s) \qquad (15b)$$

for $i,k \in S_2$, $j \in S_1$

The exercise is left to the reader. [Appendices A, B and C provide the first passage probabilities, renewal densities and the state probabilities for the ages of entrance into state i, x=20,25,30,35, 40,45 and 50.]

4. SOME SALIENT FEATURES OF THE SEMI-MARKOV MODEL

(1) First Passage Probabilities $A_{ij}(x,t)$

It is the probability that a person who enters state i at age x will make a move to state j within t time units. In the present study, the NM, W and D allow only one direct move to another transient state, namely, the PM; while the PM allows two direct moves to transient states, either to W or to D.

For an analytical example, consider the first passage probabilities from PM to D, and from D to PM for starting ages x=20,25..40. These are given in Tables 5A and 5B.

An individual who enters the PM at age 20 has a probability .016 of getting divorced by the end of 5 years; thus he enters the D at age 25 and has a probability .645 of getting back to the PM within another 5 years. On the other hand, an individual who enters the PM at age 25 has a probability .031 of getting divorced within 5 years and a probability .501 of getting remarried within another 5 years. In general, those who enter the PM at age 25 exhibit the highest probabilities of getting divorced as duration increases, but those who enter the D at age 20 exhibit the highest probabilities of getting remarried especially after 4 years of duration. And, the younger age groups between 20 and 25 entering into one or other of these two states have, in general, higher probabilities of switching from one to the other.

Looked at from the point of view of age only, those who enter the PM at age 20 have the probability .089 of getting divorced between ages 40-41, while those who enter the PM at ages 25 and 30 have only .075 and .046 probabilities respectively of getting divorced between ages 40-41. This implies that among those who get divroced between ages 40-41, those who entered the PM at an earlier age have higher probabilities. Duration spent in the PM obviously affects the probabilities of getting divorced; the longer the duration, the higher the probabilities of divorce for the individuals of the same age.

Table 5A. First Passage probabilities from the PM to D

duration t (years)	entry into the PM at age x x=15	x=20	x=25	x=30	x=35	x=40
1	0	.002	.007	.005	.005	.003
2	0	.004	.013	.011	.009	.006
3	.001	.007	.019	.016	.013	.009
4	.001	.011	.025	.021	.016	.012
5	.003	.016	.031	.026	.020	.014
6	.004	.023	.036	.031	.023	.016
7	.007	.029	.041	.035	.026	.018
8	.010	.035	.046	.038	.029	.020
9	.014	.041	.051	.042	.031	.022
10	.019	.046	.056	.046	.034	.023
11	.025	.051	.060	.048	.036	.024
12	.031	.056	.064	.051	.038	.026
13	.037	.061	.068	.054	.039	.027
14	.043	.066	.071	.056	.041	.028
15	.049	.071	.075	.059	.042	.029
16	.054	.075	.077	.061	.044	.029
17	.059	.079	.080	.063	.045	.030
18	.064	.082	.083	.064	.046	.031
19	.069	.086	.085	.066	.047	.031
20	.073	.089	.087	.067	.048	.032

Table 5B. First Passage probabilities from the D to PM

duration t (years)	entry into the D at age x x=15	x=20	x=25	x=30	x=35	x=40
1	0	.207	.221	.131	.109	.075
2	0	.334	.379	.249	.203	.141
3	0	.502	.481	.352	.298	.202
4	.095	.598	.570	.434	.358	.250
5	.126	.668	.645	.501	.413	.295
6	.307	.740	.691	.555	.456	.330
7	.418	.792	.732	.601	.495	.366
8	.565	.825	.767	.647	.530	.398
9	.649	.854	.798	.676	.557	.429
10	.710	.878	.819	.704	.583	.454
11	.773	.893	.838	.725	.603	.473
12	.818	.906	.854	.743	.624	.497
13	.847	.918	.870	.761	.642	.516
14	.812	.927	.880	.774	.660	.530
15	.893	.935	.889	.787	.674	.548
16	.906	.941	.897	.797	.686	.558
17	.918	.946	.903	.807	.699	.568
18	.928	.951	.909	.816	.710	.578
19	.936	.954	.914	.824	.718	.586
20	.943	.957	.918	.831	.728	.593

Fig.1 plots the first passage probabilities of transition from and to states of main interest, namely NM - PM, PM - D, W - PM, and D - PM, for ages of entrance into respective states of origin x=15 to 50. The curves for x=15 and x=20 almost coincide for lower durations. In general, all the curves have the same shape but for higher ages of entrance. Further examination will be done on these curves in section 5.

(2) The duration-stay probabilities and mean length of stay

The duration-stay probabilities are given by $1-A_i(x,t)$. If $D_i(x,t) = 1-A_i(x,t)$, $D_i(x,t)$ represents the probability that an individual who enters state i at age x will still be there t time periods later. Further,

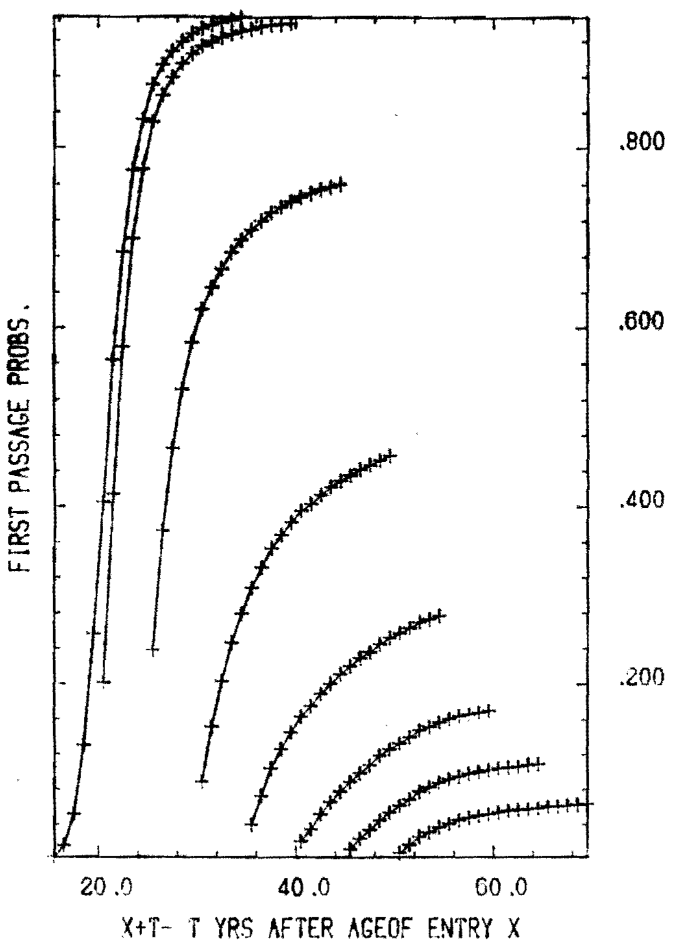
$$S_i(x,t) = D_i(x,1) + D_i(x,2) + \ldots\ldots\ldots + D_i(x,t)$$

computes the mean length of stay in state i during the time interval $(0,t]$. These values are provided in Table 6 for x=15,20 and 25 for the PM as an example.

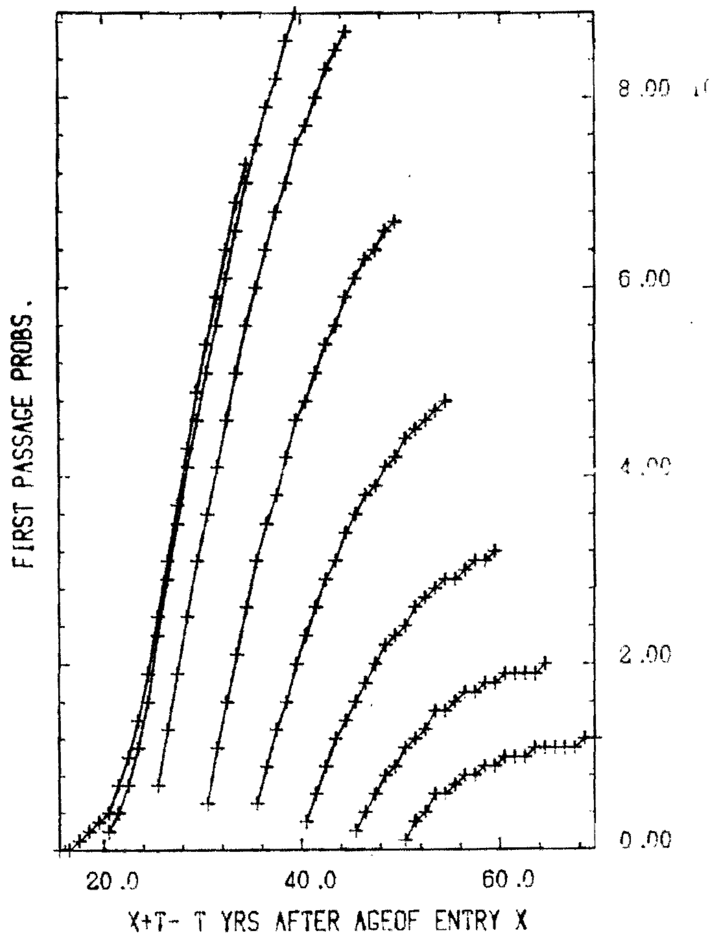
Table 6. Duration-stay probabilities (D_i) and mean length of stay (S_i) in the Present Married State

duration t (years)	entry into the PM at ages					
	x= 15		x= 20		x= 25	
	$D_2(15,t)$	$S_2(15,t)$	$D_2(20,t)$	$S_2(20,t)$	$D_2(25,t)$	$S_2(25,t$
1	1.000	1.000	.998	.998	.992	.992
2	.999	1.999	.994	1.992	.984	1.976
3	.998	2.997	.990	2.982	.976	2.952
4	.996	3.993	.985	3.967	.969	3.921
5	.984	4.987	.979	4.946	.961	4.882
6	.992	5.979	.971	5.917	.955	5.837
7	.988	6.967	.963	6.880	.948	6.785
8	.984	7.951	.956	7.836	.941	7.726
9	.979	8.930	.948	8.784	.934	8.660
10	.973	9.903	.941	9.225	.927	9.587
11	.965	10.868	.935	10.660	.920	10.507
12	.958	11.826	.928	11.588	.914	11.421
13	.950	12.776	.921	12.509	.908	12.329
14	.943	13.719	.914	13.423	.902	13.231
15	.936	14.655	.908	14.331	.895	14.126
16	.929	15.584	.901	15.232	.889	15.015
17	.922	16.506	.895	16.127	.882	15.897
18	.915	17.421	.889	17.016	.875	16.772
19	.909	18.330	.883	17.899	.869	17.641
20	.902	19.232	.876	18.775	.861	18.502

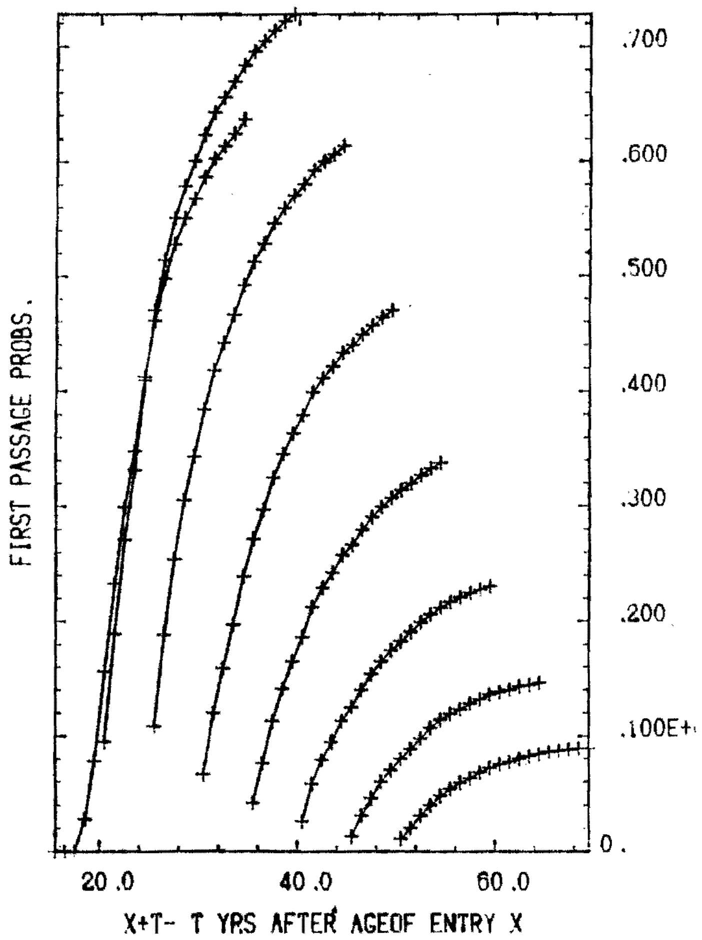
NM TO PM



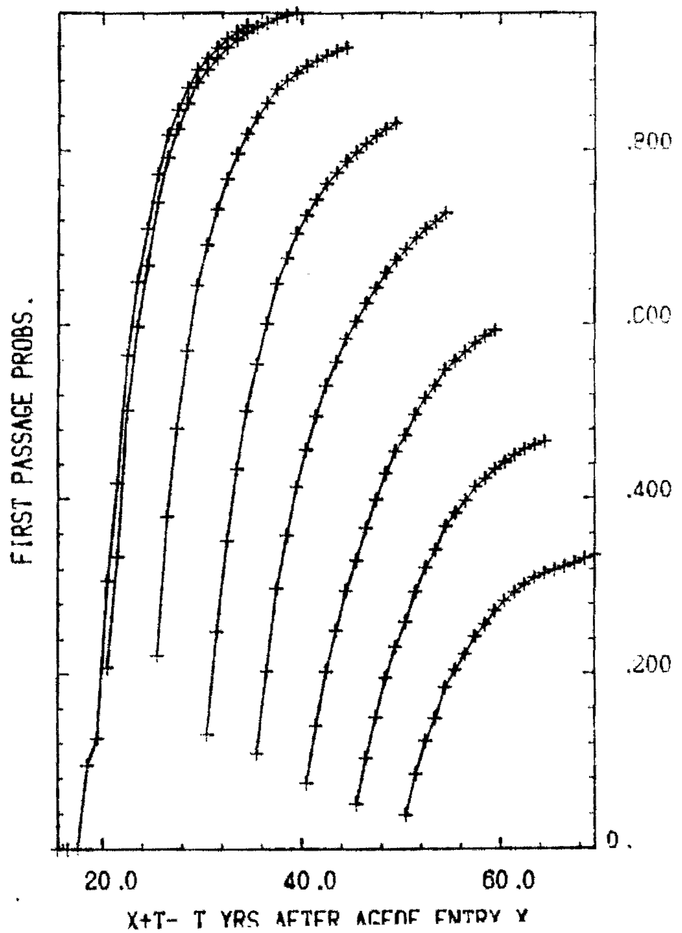
PM TO DIV



W TO PM



D TO PM



(3) Mean number of visits to transient states

The renewal densities $m_{ij}(x,t)$, when cumulated over t represents the mean number of visits to transient states j from the state of origin i during t time units. These values are provided in the following Table 7 for each state of origin i for 20 years of duration.

Table 7. Mean number of transitions to transient states within 20 years of duration

age of entry into state	from NM to			from PM to			from W to			from D to		
	PM	W	D	PM	W	D	PM	W	D	PM	W	D
15	.981	.011	.058	.047	.014	.073	.655	.008	.034	.973	.010	.056
20	.992	.018	.069	.058	.024	.092	.754	.013	.049	1.000	.019	.070
25	.786	.022	.051	.061	.034	.090	.631	.015	.034	.852	.024	.056
30	.466	.020	.019	.039	.054	.068	.478	.020	.018	.847	.037	.033
35	.280	.018	.001	.030	.091	.048	.340	.024	.008	.739	.053	.018
40	.181	.020	.000	.018	.148	.031	.233	.028	.000	.599	.075	.010
45	.110	.023	.000	.016	.242	.019	.147	.030	.000	.469	.095	.000
50	.064	.020	.000	.007	.366	.010	.091	.028	.000	.377	.106	.000

As is obvious from the table, the mean number of transitions from any state of origin to the PM and to the D shows a definite decreasing pattern for increasing ages of entrance into these states of origin. On the other hand, from any state of origin to the W they show an increasing pattern, except for some fluctuations in the case of the NM. It is worth noting also that no transitions to the D are to be found from the cohorts of the NM, W and D starting at ages of 40 or 45 (in the case of the D); all the divorces observed are experienced only by the cohort of the PM from that age of entrance onward.

(4) <u>State Probabilities</u>

It is the probability that a person who enters state i at age x will be <u>found</u> in state j within t time units. It is <u>not</u> the probability of making a move from state i to state j; before being found in state j, the person could have made multiple moves. These probabilities, as was already pointed out, form a Markov chain. And hence, they would correspond to the values of the table of the Expected Number of Survivors obtained through the status-based approach of the Markov model.

But the steps to find these state probabilities are different in the semi-Markov model in as much as they take into account not only the effects of age but also of duration. The various steps towards the construction of the state probabilities provide us with the first passage probabilities, their densities and renewal densities, all these portraying the effect of duration on transitions between states of those individuals who enter a particular state at a specific age.

Wherever direct transitions (called also "real" transitions) are possible, the first passage probabilities give the probabilities of <u>making a move</u> from one state to another within t time units. These are basic in the semi-Markov model, but not provided by the Markov model. Analytically, it is the backward equation based on the first jump which lends itself most easily to the estimation of these basic probabilities. Further, making use of the first passage densities, renewal densities are found which account for multiple and indirect transitions (called also "virtual" transitions).

Despite the labour involved, it is worth examining how the state probabilities are obtained in the semi-Markov model. The equation (17a) provides the mathematical formula for finding the state probabilities. Its interpretation is as follows: the probability that a person who enters state i at age x will be found in state j at age (x+t) is equal to the probability that he makes a move, either real or virtual, to state j within t time units (given by $m_{ij}(x,s)$), and stays in the same state for an additional t-s time units.

Thus, for example, we have $P_{12}(20,10) = 0.877$ which can be found from Table 3. This value has been obtained by

$$\begin{aligned}P_{12}(20,10) = \; & m_{12}(20,0).(1-A_2(20,10)) + m_{12}(20,1).(1-A_2(21,9)) + \\ & m_{12}(20,2).(1-A_2(22,8)) + m_{12}(20,3).(1-A_2(23,7)) + \\ & m_{12}(20,4).(1-A_2(24,6)) + m_{12}(20,5).(1-A_2(25,5)) + \\ & m_{12}(20,6).(1-A_2(26,4)) + m_{12}(20,7).(1-A_2(27,3)) + \\ & m_{12}(20,8).(1-A_2(28,2)) + m_{12}(20,9).(1-A_2(29,1)) + \\ & m_{12}(20,10).(1-A_2(30,0))\end{aligned}$$

$$\begin{aligned}= \; & 0 + (.201 * .943) + (.213 * .947) + (.164 * .951) + \\ & (.121 * .956) + (.077 * .961) + (.054 * .969) + \\ & (.032 * .977) + (.023 * .985) + (.018 * .992) + \\ & (.015 * 1.000)\end{aligned}$$

$$\begin{aligned}= \; & .1895 + .2017 + .1559 + .1157 + .0740 + \\ & .0523 + .0313 + .0226 + .0178 + .0150\end{aligned}$$

$$= .876$$

This implies that out of 877 individuals found in state 2, 190 have made their move to state 2 within one year and have stayed for nine years in state 2, 74 individuals have moved to state 2 within 5 years and have stayed in state 2 for another five years, etc.

However, since the m's are renewal densities, the passage to state 2 could have been either real or virtual. This can be further examined from the eq.(12). Thus, for example,

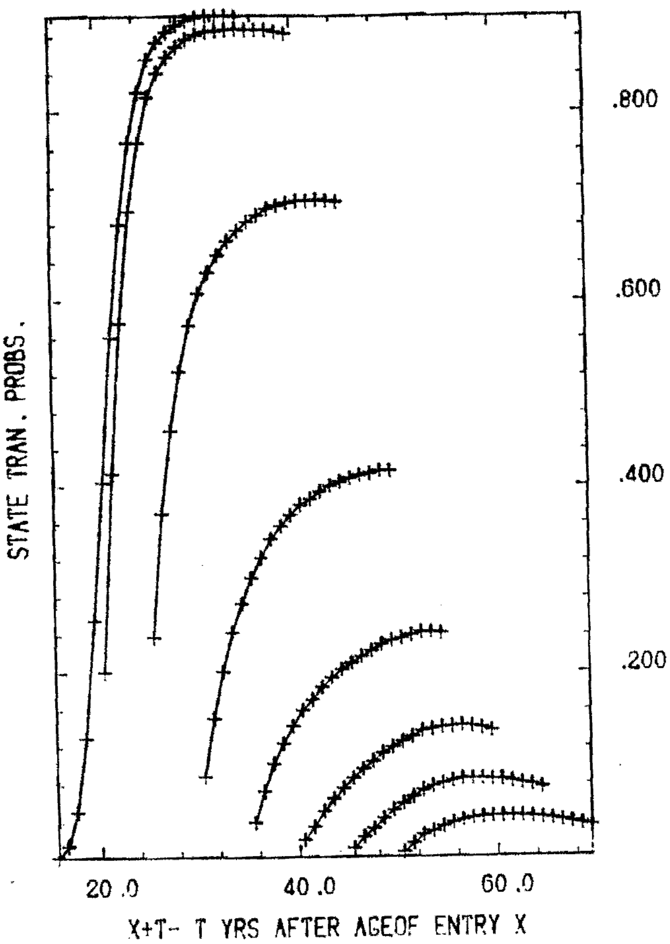
$m_{12}(20,8) = .0226$ can be seen to be composed of:

$$= m_{11}(20,0).a_{12}(20,8) + m_{13}(20,6).a_{32}(26,2) + m_{14}(20,3).a_{42}(23,5)$$
$$+ m_{13}(20,7).a_{32}(27,1) + m_{14}(20,4).a_{42}(24,4)$$
$$+ m_{14}(20,5).a_{42}(25,3)$$
$$+ m_{14}(20,6).a_{42}(26,2)$$
$$+ m_{14}(20,7).a_{42}(27,1)$$

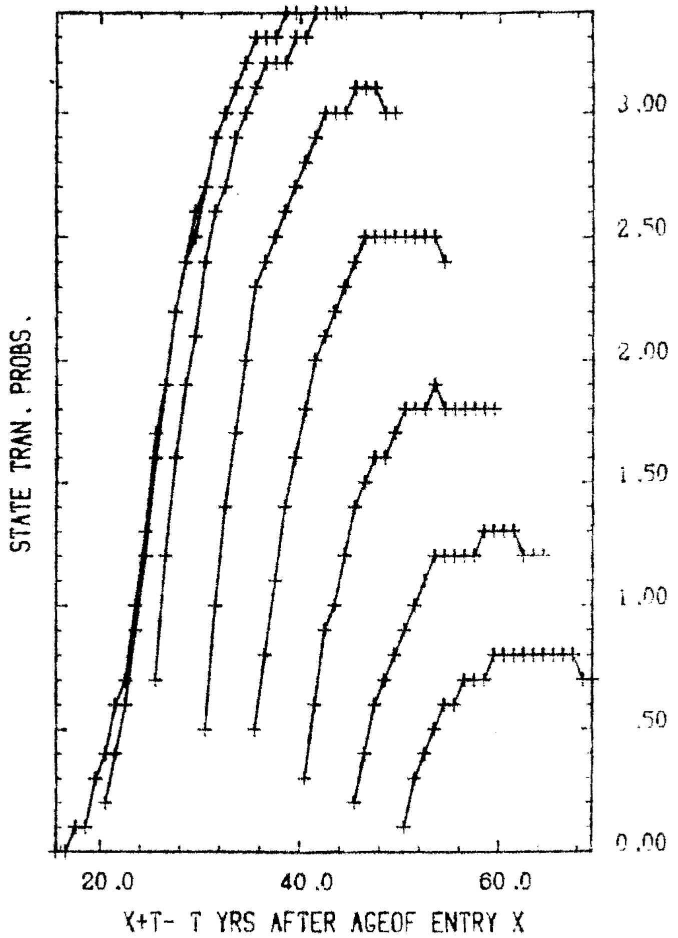
where more terms (not necessarily a greater number of cases) are coming from the state 4(D). This kind of analysis can be carried on to the point, where one finally arrives at the first passage probabilities.

Note that the state probability matrices are stochastic matrices. Fig.2 plots these state probabilities for states of main interest. If fig.1 of first passage probabilities is laid over fig.2 of state probabilities, one notices that the curves in both figures coincide except for the upper tail-ends of state probability curves and except for transitions from the PM to the D. They seem to be similar in shape, but differ in their levels. This seems to indicate that the study of state probabilities is perhaps better effected through the study of first passage probabilities; because the latter are the probabilities of making a move from one state to another while the former are the probabilities of being found in a specific state.

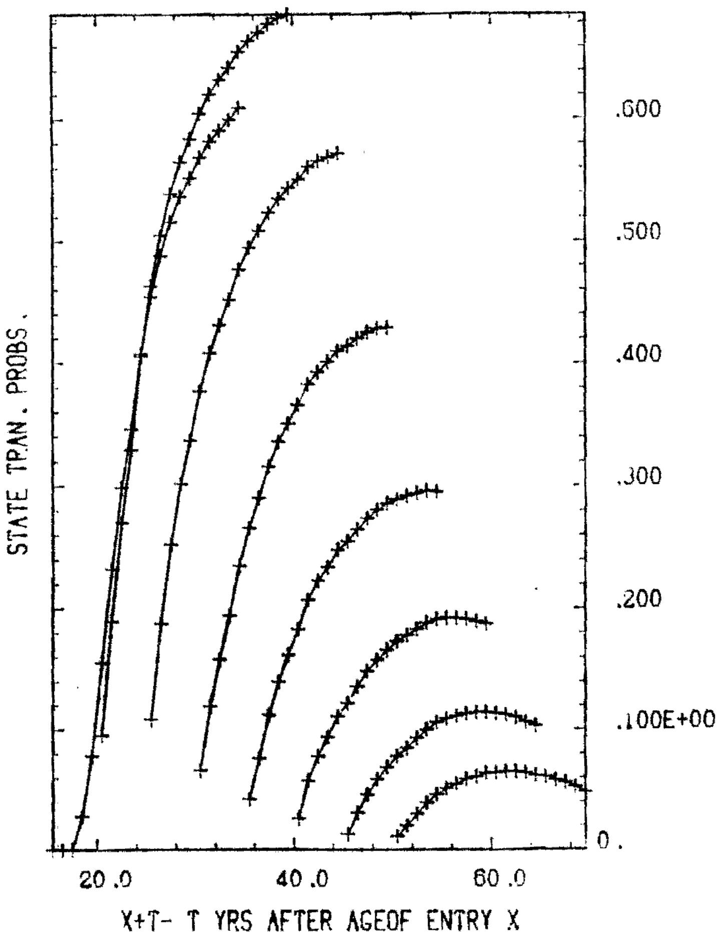
NM TO PM



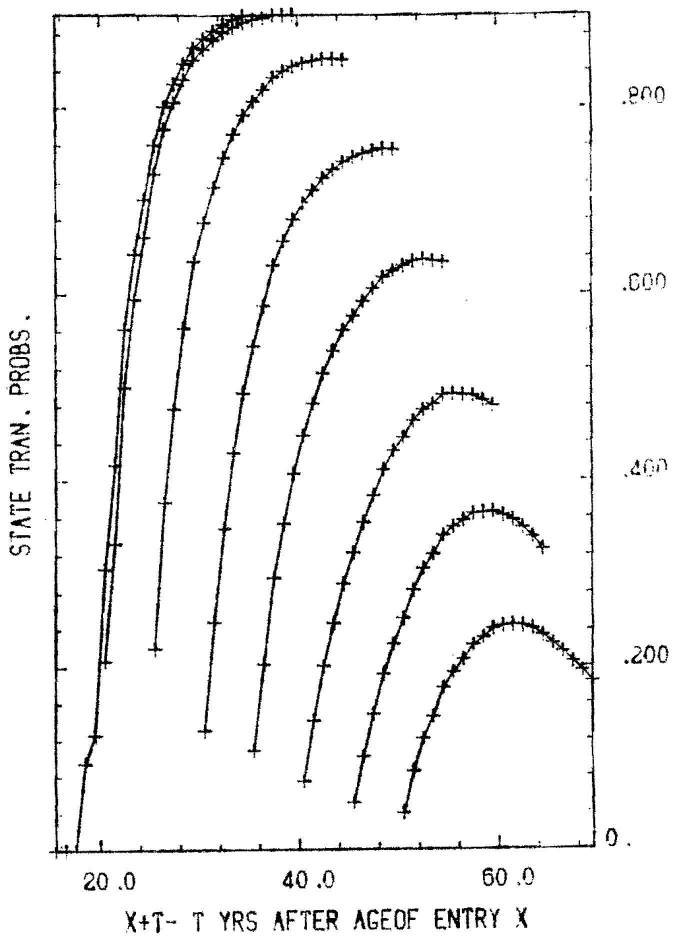
PM TO DIV



W TO PM



D TO PM



PART II

5. PARAMETRIC FORMS OF THE ONE-STEP TRANSITION PROBABILITIES

One of the advantages of the semi-Markov model is that it facilitates a parametrization of its basic probabilities, namely the first passage probabilities, unlike the Markov model with respect to state probabilities.

The first passage probabilities can be expressed in a parametric form by a proper choice of density function. In general,

$$a_{ij}(x,t) = \Pi_{ij}(x).f\left[\alpha(x), \beta(x),....; t\right]$$

where f is a density function with the parameters $\alpha(x)$, $\beta(x)$... and t. These parameters can be estimated through various techniques at our disposal. Computer programs are now available to estimate the parameters by the method of Maximum Likelihood or through the use of the Minimization Principle; for example, the CERN and NAG computer programs. But one is handicapped in making use of these computer programs because of the lack of knowledge about the limits of these parameters.

As the case under study is the process of entry into and exit from marriage, the model proposed by Gudmund Hernes (1972) was tried. This model has been constructed to capture only the process of entry into first marriage and has been built on quite interesting sociological considerations of two main forces influencing the unmarried. The first force is the increase in social pressure on a single person that accompanies the increase in the percentage of the cohort already married - the cohort to which he or she belongs. Thus, social pressure to marry is taken to be proportional to the percentage of the cohort already married, and the rate of change in the probability of getting married is taken to be proportional to this pressure. The second force is marriageability which generally declines with age. These two forces have opposite effects; one increases the pressure to marry, the other reduces the capacity to marry.

The final form of the Hernes' model is given simply by

$$P_t = \frac{1}{1 + \frac{1}{ka^{b^t}}} \qquad (18)$$

where P_t is the proportion of the cohort already married at time t, $\log a = \frac{A}{\log b}$, A is the average initial marriageability, b (<1) is the constant of deterioration in marriageability, and $k = \frac{P_0}{a(1-P_0)}$. If we have the estimates of k, a and b, then P_0 and A can be calculated and the model can be completely specified. This model has a special relevance of application to the case under study because it can be viewed as describing a non-homogeneous diffusion process.

Before the application of this model, certain points are to be borne in mind.

i) P_0 is well defined and is not equal to zero, because from (16) it can be seen that $P_0 = \frac{1}{1+\frac{1}{ka}}$. Practically speaking, this means that in fitting the data, the first year of the process should be taken to be t_0, that is, 0.

ii) The form given in (16) looks like a logistic but its inflection point is not midway between 0 and its upper asymptote, so that the limbs of the curve are not symmetric about the inflection point as the logistic is.

iii) The asymptote of the curve is given by $\underset{t\to\infty}{Lt} P_t = \frac{1}{1+\frac{1}{k}}$ as $b < 1$.

iv) If we let $g_t = ka^{b^t}$, then g_t is a Gompertz function and the parameters a, b, and k can be estimated by the usual method of selected points (3-points procedure), by dividing the data into three equal sections. Then the estimates are given by the formulae :

$$b^T = \frac{\Sigma_3 \log g_t - \Sigma_2 \log g_t}{\Sigma_2 \log g_t - \Sigma_1 \log g_t} \qquad (17)$$

$$\log a = (\Sigma_2 \log g_t - \Sigma_1 \log g_t). \frac{b - 1}{(b^T-1)^2} \qquad (18)$$

$$\log k = \frac{1}{T} (\Sigma_1 \log g_t - \frac{(b^T-1)}{(b -1)} .\log a) \qquad (19)$$

where Σ_i denotes the sum of logarithms of the observed cumulative percentages of the i-th section and T is the number of observations in each section.

The first passage probabilities $A_{ij}(x,t)$ are nothing else but the cumulative distribution,as t increases, of the first passage densities $a_{ij}(x,t)$. Therefore, this model can be applied to fit the values of $A_{ij}(x,t)$ for each x, i=NM and j=PM. With 24 observations, the first passage probabilites have been fitted, and they are presented in Table 8 . The fit is remarkably good, remarkable in the light of unsatisfactory fits attempted with many other distributions like gamma, log-normal and even logistic, through the Minimization Principle. In the table, ALPHA stands for the parameter "a", BETA for "b" and KAPPA for "k", ABILITY for "A" which is the average initial marriageability.

The average initial marriageability is highest at age 15 and decreases up to age 35, it then moves upward till 45, and once again falls down from age 50. The coefficient b, the constant of deterioration in marriageability, fluctuates. The asymptotes decrease throughout.

Table 8.

FITTED FIRST PASSAGE PROBS. -NM TO PM
OBSERVED VALUES IN BRACKETS   GOMPERTZ 3 POINTS FIT

	AGE(15)	AGE(20)	AGE(25)	AGE(30)	AGE(35)	AGE(40)	AGE(45)	AGE(50)
ALPHA	.000111	.021742	.131874	.156647	.152805	.123335	.127466	.193936
BETA	.815798	.814673	.833679	.847558	.859413	.835104	.800980	.819028
KAPPA	21.617324	17.071747	3.394197	.918645	.425205	.219328	.125977	.070926
ABILITY	1.853771	.784728	.368526	.306604	.284618	.377130	.457133	.327450
ASYMPT	.955786	.944665	.772427	.478799	.298347	.179876	.111883	.066229
1	.002 (.003)	.271 (.201)	.309 (.238)	.126 (.089)	.061 (.040)	.026 (.021)	.016 (.012)	.014 (.008)
2	.013 (.016)	.430 (.414)	.385 (.373)	.160 (.152)	.078 (.073)	.037 (.035)	.024 (.024)	.018 (.017)
3	.048 (.052)	.574 (.579)	.454 (.466)	.195 (.203)	.096 (.104)	.048 (.052)	.033 (.034)	.023 (.026)
4	.134 (.130)	.683 (.700)	.512 (.532)	.229 (.246)	.114 (.126)	.061 (.066)	.042 (.045)	.028 (.031)
5	.277 (.256)	.760 (.776)	.561 (.584)	.261 (.279)	.132 (.145)	.073 (.078)	.051 (.054)	.033 (.037)
6	.446 (.405)	.812 (.829)	.600 (.621)	.290 (.308)	.150 (.163)	.086 (.089)	.060 (.062)	.037 (.041)
7	.596 (.564)	.848 (.859)	.632 (.646)	.316 (.331)	.166 (.175)	.097 (.099)	.068 (.069)	.041 (.045)
8	.708 (.685)	.873 (.879)	.658 (.667)	.339 (.353)	.182 (.189)	.108 (.108)	.075 (.078)	.045 (.048)
9	.784 (.775)	.890 (.894)	.679 (.685)	.359 (.368)	.196 (.200)	.118 (.119)	.082 (.083)	.048 (.050)
10	.834 (.832)	.903 (.905)	.696 (.699)	.377 (.382)	.208 (.211)	.127 (.126)	.087 (.088)	.051 (.052)
11	.868 (.871)	.912 (.914)	.710 (.710)	.392 (.395)	.220 (.220)	.134 (.133)	.091 (.092)	.054 (.054)
12	.891 (.893)	.920 (.919)	.721 (.720)	.405 (.403)	.230 (.229)	.141 (.140)	.095 (.095)	.056 (.056)
13	.907 (.908)	.925 (.924)	.730 (.729)	.416 (.413)	.239 (.236)	.147 (.148)	.098 (.098)	.058 (.057)
14	.919 (.919)	.927 (.928)	.737 (.735)	.425 (.422)	.247 (.245)	.152 (.152)	.101 (.100)	.059 (.058)
15	.927 (.928)	.932 (.931)	.743 (.741)	.433 (.429)	.253 (.252)	.156 (.157)	.103 (.102)	.060 (.059)
16	.934 (.934)	.935 (.933)	.748 (.746)	.440 (.435)	.259 (.257)	.160 (.161)	.105 (.104)	.061 (.061)
17	.938 (.938)	.937 (.936)	.752 (.750)	.446 (.442)	.265 (.263)	.163 (.164)	.106 (.105)	.062 (.061)
18	.942 (.941)	.938 (.938)	.756 (.754)	.451 (.447)	.269 (.270)	.166 (.166)	.107 (.106)	.063 (.062)
19	.945 (.944)	.939 (.939)	.759 (.757)	.455 (.453)	.273 (.273)	.168 (.168)	.108 (.108)	.063 (.063)
20	.947 (.947)	.940 (.940)	.761 (.760)	.459 (.458)	.277 (.277)	.170 (.170)	.109 (.109)	.064 (.064)
21	.949 (.949)	.941 (.941)	.763 (.763)	.462 (.462)	.280 (.280)	.172 (.171)	.109 (.110)	.064 (.064)
22	.750 (.950)	.942 (.942)	.765 (.766)	.464 (.466)	.282 (.283)	.173 (.173)	.110 (.110)	.065 (.065)
23	.951 (.952)	.942 (.943)	.766 (.768)	.467 (.471)	.285 (.285)	.174 (.174)	.110 (.111)	.065 (.066)
24	.952 (.953)	.943 (.944)	.767 (.770)	.469 (.474)	.286 (.286)	.175 (.175)	.111 (.112)	.065 (.067)

FITTED FIRST PASSAGE PROBS. -PM TO DIV
OBSERVED VALUES IN BRACKETS   GOMPERTZ 3 POINTS FIT

	AGE(15)	AGE(20)	AGE(25)	AGE(30)	AGE(35)	AGE(40)	AGE(45)	AGE(50)
ALPHA	.005779	.021713	.089497	.100283	.124787	.127873	.131778	.162680
BETA	.857876	.855193	.863723	.852606	.848079	.832797	.805795	.813701
KAPPA	.115794	.119458	.111060	.080851	.055122	.034916	.020701	.012321
ABILITY	.789882	.599091	.353594	.366714	.342933	.376308	.437603	.374383
ASYMPT	.103938	.106711	.099959	.074803	.052242	.033738	.020282	.012171
1	0.000 (0.000)	.003 (.002)	.010 (.007)	.008 (.005)	.007 (.005)	.004 (.003)	.003 (.002)	.002 (.001)
2	0.000 (0.000)	.004 (.004)	.014 (.013)	.011 (.011)	.009 (.009)	.006 (.006)	.004 (.004)	.003 (.003)
3	.003 (.001)	.007 (.007)	.018 (.019)	.015 (.016)	.012 (.013)	.008 (.009)	.006 (.006)	.004 (.004)
4	.004 (.001)	.011 (.011)	.023 (.025)	.019 (.021)	.015 (.016)	.011 (.012)	.007 (.008)	.005 (.006)
5	.007 (.003)	.015 (.016)	.028 (.031)	.023 (.026)	.018 (.020)	.013 (.014)	.009 (.009)	.006 (.006)
6	.010 (.004)	.020 (.023)	.034 (.036)	.028 (.031)	.022 (.023)	.015 (.016)	.010 (.011)	.006 (.007)
7	.015 (.007)	.026 (.029)	.039 (.041)	.032 (.035)	.025 (.026)	.017 (.018)	.012 (.012)	.007 (.008)
8	.020 (.010)	.032 (.035)	.045 (.046)	.037 (.038)	.028 (.029)	.019 (.020)	.013 (.013)	.008 (.008)
9	.025 (.014)	.038 (.041)	.050 (.051)	.041 (.042)	.031 (.031)	.021 (.022)	.014 (.015)	.009 (.009)
10	.031 (.019)	.045 (.046)	.055 (.056)	.045 (.046)	.033 (.034)	.023 (.023)	.015 (.015)	.009 (.009)
11	.037 (.025)	.051 (.051)	.060 (.060)	.048 (.048)	.036 (.036)	.024 (.024)	.016 (.016)	.010 (.010)
12	.043 (.031)	.057 (.056)	.064 (.064)	.052 (.051)	.038 (.038)	.026 (.026)	.017 (.017)	.010 (.010)
13	.049 (.037)	.062 (.061)	.068 (.068)	.054 (.054)	.040 (.039)	.027 (.027)	.017 (.017)	.010 (.010)
14	.054 (.043)	.067 (.066)	.072 (.071)	.057 (.056)	.041 (.041)	.028 (.028)	.018 (.018)	.011 (.011)
15	.060 (.049)	.072 (.071)	.075 (.075)	.059 (.059)	.043 (.042)	.029 (.029)	.018 (.018)	.011 (.011)
16	.065 (.054)	.076 (.075)	.078 (.077)	.061 (.061)	.044 (.044)	.030 (.029)	.019 (.019)	.011 (.011)
17	.069 (.059)	.080 (.079)	.081 (.080)	.063 (.063)	.045 (.045)	.030 (.030)	.019 (.019)	.011 (.011)
18	.073 (.064)	.084 (.082)	.083 (.083)	.065 (.064)	.046 (.046)	.031 (.031)	.019 (.019)	.012 (.011)
19	.077 (.069)	.087 (.086)	.085 (.085)	.066 (.066)	.047 (.047)	.031 (.031)	.019 (.019)	.012 (.012)
20	.081 (.073)	.089 (.089)	.087 (.087)	.067 (.067)	.048 (.048)	.032 (.032)	.020 (.020)	.012 (.012)
21	.084 (.078)	.092 (.072)	.089 (.089)	.069 (.068)	.049 (.049)	.032 (.032)	.020 (.020)	.012 (.012)
22	.086 (.081)	.094 (.095)	.090 (.091)	.069 (.070)	.049 (.049)	.032 (.032)	.020 (.020)	.012 (.012)
23	.089 (.085)	.096 (.097)	.092 (.092)	.070 (.071)	.050 (.050)	.033 (.033)	.020 (.020)	.012 (.012)
24	.091 (.088)	.097 (.079)	.093 (.094)	.071 (.072)	.050 (.050)	.033 (.033)	.020 (.020)	.012 (.012)

Table 8. contd.

FITTED FIRST PASSAGE PROBS. -DIV TO PM
OBSERVED VALUES IN BRACKETS GOMPERTZ 3 POINTS FIT

	AGE(15)	AGE(20)	AGE(25)	AGE(30)	AGE(35)	AGE(40)	AGE(45)	AGE(50)
ALPHA	.002994	.008875	.023798	.035735	.058349	.068027	.081622	.119335
BETA	.884381	.896215	.893692	.882548	.886363	.868042	.838689	.826994
KAPPA	37.817548	40.350963	17.516756	6.765108	3.552138	1.748194	.953724	.539996
ABILITY	.714000	.517694	.420146	.416260	.342746	.380370	.440783	.403815
ASYMPT	.974238	.975817	.945995	.871219	.780323	.636125	.488157	.350648
1	0.000 (0.000)	.264 (.207)	.294 (.221)	.195 (.131)	.172 (.109)	.106 (.075)	.072 (.051)	.061 (.039)
2	0.000 (0.000)	.369 (.334)	.383 (.379)	.263 (.249)	.223 (.203)	.145 (.141)	.104 (.103)	.085 (.085)
3	0.000 (0.000)	.476 (.502)	.469 (.481)	.336 (.352)	.276 (.298)	.187 (.202)	.141 (.150)	.112 (.123)
4	.404 (.095)	.574 (.578)	.549 (.570)	.407 (.434)	.329 (.358)	.232 (.250)	.179 (.195)	.140 (.149)
5	.519 (.126)	.657 (.668)	.617 (.645)	.473 (.501)	.381 (.413)	.275 (.295)	.216 (.231)	.167 (.184)
6	.620 (.307)	.724 (.740)	.675 (.691)	.532 (.555)	.429 (.456)	.317 (.330)	.252 (.260)	.192 (.204)
7	.701 (.418)	.777 (.792)	.723 (.732)	.584 (.601)	.472 (.495)	.356 (.366)	.285 (.294)	.215 (.222)
8	.764 (.565)	.818 (.825)	.762 (.767)	.628 (.647)	.512 (.530)	.392 (.398)	.315 (.322)	.235 (.243)
9	.811 (.649)	.850 (.854)	.793 (.796)	.665 (.676)	.546 (.557)	.424 (.429)	.341 (.342)	.253 (.257)
10	.847 (.710)	.874 (.878)	.818 (.819)	.696 (.704)	.576 (.583)	.452 (.454)	.363 (.368)	.269 (.272)
11	.873 (.773)	.893 (.893)	.839 (.838)	.722 (.725)	.603 (.603)	.476 (.473)	.383 (.383)	.282 (.283)
12	.894 (.818)	.907 (.906)	.855 (.854)	.744 (.743)	.626 (.624)	.498 (.497)	.399 (.397)	.293 (.293)
13	.909 (.847)	.919 (.918)	.869 (.870)	.763 (.761)	.646 (.642)	.517 (.516)	.413 (.413)	.303 (.302)
14	.921 (.872)	.928 (.927)	.880 (.880)	.778 (.774)	.663 (.660)	.533 (.530)	.425 (.423)	.311 (.310)
15	.930 (.893)	.936 (.735)	.890 (.889)	.791 (.787)	.678 (.674)	.547 (.548)	.435 (.434)	.318 (.315)
16	.938 (.906)	.942 (.941)	.898 (.897)	.802 (.797)	.690 (.686)	.559 (.558)	.444 (.442)	.323 (.319)
17	.944 (.918)	.947 (.946)	.904 (.903)	.812 (.807)	.702 (.699)	.569 (.568)	.451 (.450)	.328 (.323)
18	.948 (.928)	.951 (.951)	.910 (.909)	.820 (.816)	.711 (.710)	.578 (.578)	.457 (.457)	.332 (.327)
19	.952 (.936)	.954 (.954)	.914 (.914)	.826 (.824)	.720 (.718)	.586 (.586)	.462 (.462)	.335 (.332)
20	.956 (.943)	.957 (.957)	.918 (.918)	.832 (.831)	.727 (.728)	.593 (.593)	.466 (.466)	.338 (.336)
21	.958 (.948)	.960 (.960)	.922 (.922)	.837 (.837)	.734 (.734)	.599 (.599)	.470 (.469)	.340 (.340)
22	.961 (.953)	.962 (.962)	.925 (.925)	.842 (.844)	.739 (.740)	.604 (.604)	.473 (.472)	.342 (.344)
23	.962 (.957)	.964 (.964)	.927 (.928)	.845 (.849)	.744 (.746)	.608 (.609)	.475 (.475)	.343 (.348)
24	.964 (.960)	.965 (.965)	.930 (.931)	.849 (.853)	.748 (.750)	.612 (.612)	.477 (.479)	.345 (.352)

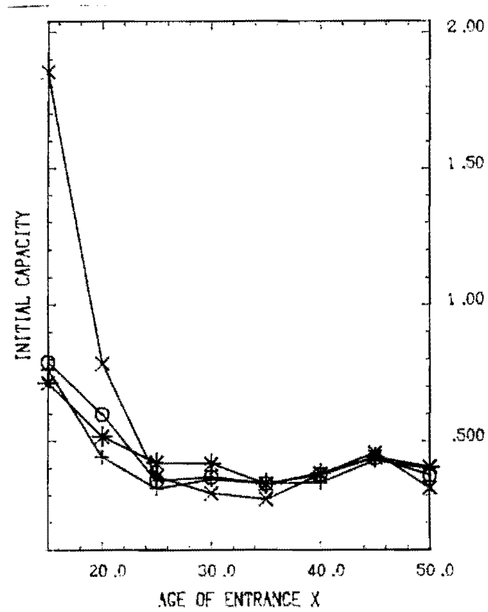
FITTED FIRST PASSAGE PROBS. -WID TO PM
OBSERVED VALUES IN BRACKETS GOMPERTZ 3 POINTS FIT

	AGE(15)	AGE(20)	AGE(25)	AGE(30)	AGE(35)	AGE(40)	AGE(45)	AGE(50)
ALPHA	.022165	.051203	.094355	.097995	.115501	.130544	.118002	.153062
BETA	.818584	.861852	.871385	.856923	.852932	.843417	.818512	.808642
KAPPA	2.070183	3.211043	1.889702	1.006491	.566130	.325762	.179801	.101325
ABILITY	.762528	.441845	.325000	.358664	.343362	.346727	.427982	.398654
ASYMPT	.674286	.762529	.653944	.501618	.361484	.245717	.152399	.092002
1	0.000 (0.000)	.141 (.095)	.151 (.108)	.090 (.066)	.061 (.042)	.041 (.026)	.021 (.013)	.015 (.011)
2	0.000 (0.000)	.199 (.189)	.195 (.188)	.121 (.120)	.082 (.076)	.055 (.058)	.030 (.031)	.022 (.020)
3	0.000 (0.000)	.261 (.271)	.239 (.254)	.155 (.159)	.105 (.113)	.071 (.079)	.041 (.046)	.029 (.031)
4	.204 (.028)	.324 (.331)	.284 (.305)	.189 (.197)	.129 (.141)	.088 (.095)	.053 (.060)	.036 (.040)
5	.272 (.078)	.384 (.409)	.326 (.343)	.223 (.239)	.153 (.165)	.104 (.113)	.064 (.070)	.043 (.048)
6	.338 (.156)	.439 (.470)	.366 (.384)	.256 (.271)	.176 (.186)	.120 (.125)	.076 (.080)	.050 (.054)
7	.397 (.233)	.487 (.514)	.402 (.418)	.286 (.296)	.198 (.212)	.135 (.140)	.086 (.089)	.057 (.059)
8	.448 (.299)	.529 (.551)	.434 (.442)	.314 (.324)	.218 (.229)	.149 (.154)	.096 (.098)	.062 (.063)
9	.490 (.348)	.565 (.579)	.463 (.466)	.339 (.345)	.236 (.242)	.162 (.165)	.105 (.107)	.067 (.068)
10	.525 (.412)	.596 (.601)	.488 (.492)	.361 (.363)	.253 (.257)	.173 (.175)	.112 (.114)	.071 (.072)
11	.553 (.461)	.621 (.624)	.510 (.512)	.380 (.379)	.267 (.266)	.184 (.183)	.119 (.119)	.075 (.075)
12	.576 (.498)	.643 (.643)	.529 (.528)	.397 (.399)	.280 (.279)	.192 (.191)	.124 (.123)	.078 (.077)
13	.595 (.528)	.661 (.656)	.546 (.546)	.412 (.411)	.291 (.290)	.200 (.199)	.129 (.128)	.080 (.080)
14	.610 (.551)	.676 (.670)	.560 (.559)	.424 (.421)	.301 (.299)	.207 (.206)	.133 (.132)	.083 (.082)
15	.622 (.568)	.689 (.684)	.573 (.570)	.435 (.433)	.310 (.307)	.213 (.212)	.136 (.136)	.084 (.084)
16	.631 (.587)	.700 (.696)	.583 (.580)	.445 (.440)	.317 (.314)	.218 (.217)	.139 (.138)	.086 (.086)
17	.639 (.603)	.709 (.705)	.593 (.592)	.453 (.449)	.323 (.320)	.222 (.221)	.142 (.140)	.087 (.087)
18	.646 (.614)	.717 (.714)	.601 (.600)	.460 (.457)	.329 (.327)	.225 (.225)	.143 (.143)	.088 (.088)
19	.651 (.625)	.724 (.722)	.608 (.606)	.466 (.464)	.334 (.333)	.229 (.228)	.145 (.144)	.089 (.089)
20	.655 (.637)	.729 (.728)	.614 (.614)	.471 (.470)	.338 (.338)	.231 (.231)	.146 (.146)	.089 (.089)
21	.659 (.646)	.734 (.734)	.619 (.618)	.475 (.475)	.341 (.342)	.233 (.233)	.147 (.148)	.090 (.090)
22	.662 (.653)	.738 (.740)	.624 (.624)	.479 (.480)	.344 (.345)	.235 (.236)	.148 (.149)	.090 (.090)
23	.664 (.661)	.741 (.745)	.628 (.629)	.482 (.485)	.347 (.348)	.237 (.238)	.149 (.150)	.091 (.090)
24	.666 (.667)	.744 (.748)	.631 (.634)	.485 (.490)	.349 (.351)	.238 (.239)	.150 (.151)	.091 (.091)

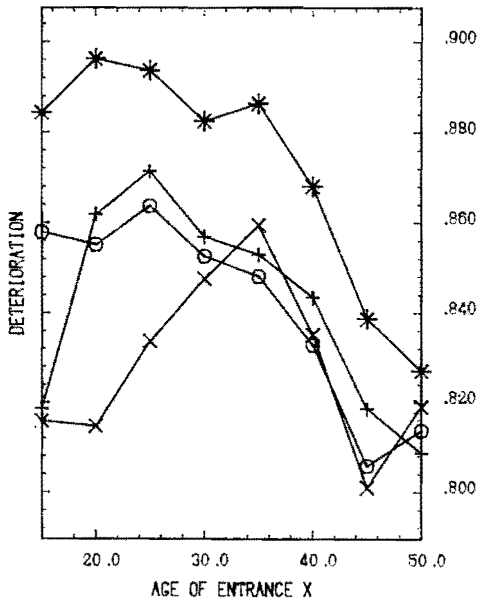
Encouraged by these results, the same model was thought of for fitting the first passage probabilities for remarriage and for divorce as well, on the supposition that the same or similar sociological forces are at work. Marriageability will be interpreted then as "remarriageability" or "divorceability" as the case would require. Thus, for example, the interpretation would be, in the case of transition from PM to D : a social pressure operates on the present married to get divorced, when many of their cohort are already divorced - "He or she, why not me?" attitude! And this pressure is negatively countered by the age of the individuals. Leaving aside the questions that can arise from these sociological interpretations, the fits are found once again to be good, except for the youngest cohort starting from age 15 and for some overestimates in other cohorts for the first duration interval (0,1]. These fits are also given in Table 8.

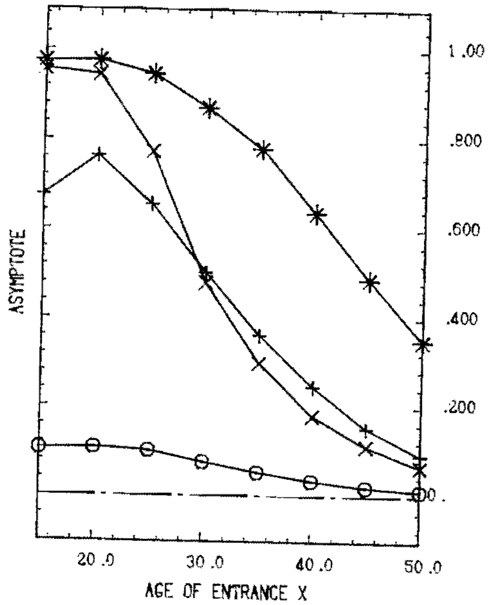
These estimated parameters a, b and k are plotted for the four main transitions + NM-PM, PM-D, W-PM and D-PM. (Fig.3).

x——x ---- NM-PM
o——o ---- PM-D
—— ---- W -PM
+——+ ---- D -PM



The initial capacity (for marriage of the NM, for remarriage of the W and the D, and for divorce of the PM) seems to almost coincide for all the cohorts from age 25 onwards. The constant of deterioration is the highest for all ages in the case of transition from the W to PM and lowest for transition from the D to the PM except for age 35. In contrast, the





asymptote is the highest in the case of transition from the W to the PM for all ages, while it is the lowest in the case of transition from the D to the PM.

6. FURTHER WORKS ENVISAGED AND CONCLUSION

Of a few suggestions put forward to relax the assumptions of homogeneity and Markovian condition inherent in the construction of multistate life tables currently in use, that of Mode has been found to be the most helpful. His suggestion to construct a semi-Markov model by extending the backward differential equations to include sojourn times in states makes feasible a computer algorithm. This algorithm winds its way through first passage probabilities and renewal densities to express the state probabilities in terms of duration spent in states and of pulls and pushes among states. In fact, the first passage probabilities have been found to present a more relevant and more realistic picture than the state probabilities.

That the semi-Markov model constructed on the methodology proposed by Mode relaxes the Markovian assumption by introducing sojourn times in states is quite clear. But that it also helps in studying the effects of heterogeneity is not that obvious. In fact, we have seen that the first passage probabilities can be parametrized. Once the parametrization is made possible, we can use these parameters in turn to study the effects of heterogeneity.

In general, if there is a vector $\underline{Z}$ of n covariates such that $\underline{Z} = (z_1, z_2, \ldots z_n)$, this vector can be taken into the parametric form of the first passage probabilities, and the parameters can be made to be dependent on the vector of covariates. For example, one of the parameters we have used in the last section, say "a", can be expressed as $a(x, \underline{Z}) = \exp(\Sigma y_r z_r)$ where y_r are the parameters of heterogeneity (of covariates) to be estimated.

In an effort at parametrizing the first passage probabilities, we found that the Hernes' model accounts well not only for the sociological forces in operation behind the process of first marriage as it was originally intended, but also those influencing the processes of remarriage and divorce as well. Now, we can bring in a greater degree of heterogeneity in the calculation of the first passage probabilities by taking account of the three culturally distinct regions in Belgium, namely Bruxelles (Brabant), Wallonia and Flanders. If dummies were to be used, these three regions have to be expressed in two dummies (say, z_1 for Wallonia, z_2 for Flanders, both in reference to Bruxelles). Further, if sex also were to be introduced, another dummy (say Z_3) can be taken for males or females, and so on. These possibilities of further heterogenization will be explored in future works.

Similarly, extending the study from 1970 to 1981, when the last census in Belgium was held, can also be done to examine the trends in transitions between marital states. If data were available, another topic of interest which is gaining attention of demographers, namely cohabitation before marriage, can as well be introduced instead of the usual four marital states.

The semi-Markov model opens new vistas for further research works which attempt to study the effects of inhomogeneities other than duration in demographic transitions.

ooooooooo

BIBLIOGRAPHY

Blumen, Kogan and McCarthy,(1955),The Industrial Mobility of Labour as Probability Process, Cornell Studies of Industry and Labour Relations, 4 Ithaca, N.Y. 19.

Çinlar E.(1975),Introduction to Stochastic Processes, Prentice Hall, N.J.

Elandt-Johnson R.C. & Johnson N.L.,(1980),Survival Models and Data Analysis, Wiley Pub., N.Y.

Feller W.,(1950), An Introduction to Probability Theory and Its Applications, Vol.I, John Wiley, London.

Feller W.,(1964), "On Semi-Markov Processes", Proceedings of the National Academy of Sciences, 51(1964)653-659.

Feller W.,(1966), An Introduction to Probability Theory and Its Applications, Vol.II, Wiley, N.Y.

Ginsberg R.B.,(1971),"Semi-Markov Processes and Mobility", Journal of Mathematical Sociology, 1(1971)236seq.

Hannan M.T.,(1982), "Multistate Demography and Event-History Analysis", WP-82-50, IIASA, Laxenburg.

Hastings D.W. & Robinson J.G.,(1973), "A Reexamination of Hernes' Model on the Process of Entry Into First Marriage for United States Women, Cohorts 1891-1945", American Sociological Review, 38(1973)138-142.

Hennessey J.C., (1980), "An Age-Dependent Absorbing Semi-Markov Model of Work Histories of the Disabled",Mathematical Bio-Sciences, 51(1980)283-304.

Hernes G., (1972), "The Process of Entry Into First Marriage", American Sociological Review, 37(1972)173-182.

Hoem J.M., (1972), "Inhomogeneous Semi-Markov Processes, Select Actuarial Tables and Duration Dependence in Demography", Population Dynamics, ed. by T.N.E.Greville, Academy Press, N.Y., pp.251-296.

Keyfitz N.,(1980), Multidimensionality in Population Analysis, IIASA, RR-80-33.

Krishnamoorthy S., (1979), "Classical Approach to Increment-decrement Life Tables, An Application to the Study of Marital Status of United States Females, 1970", Mathematical Bio-Sciences, 44(1979)139-154.

Ledent J., (1978), Some Methodological and Empirical Considerations in the Construction of Increment-decrement Life Tables, IIASA, RM-78-25.

Ledent J., (1980), "Multistate (Increment-decrement) Life Tables; Movement versus Transition Perspectives", Environment and Planning A, 12(1980)533-562.

Mode C.J. and Pickens G., (1979), "An Analysis of Postpartum Contraceptive Strategies - Policies for Preventive Medicine", Mathematical Biosciences, 47(1979)91-113.

Mode C.J. and Soyka M.G., (1980), "Linking Semi-Markov Processes in Time Series - An Approach to Longitudinal Data Analysis", Mathematical Biosciences, 51(1980)141-164.

Mode C.J., (1982), "Increment-decrement Life Tables and semi-Markovian Processes from a Sample Path Perspective", in Land K.C. & Rogers A.,(ed.), 1982.

Parzen E., (1962), Stochastic Processes, Holden Day, San Francisco, Ch.VI.

Pickens G., (1979), "A Stochastic Model of Natural Marital Fertility: Theory and Analysis of some Historical Population Data", Math. Biosciences, 48(1980)129-151.

Rogers A., (1975), Introduction to Multiregional Mathematical Demography, John-Wiley, N.Y.

Rogers A. (ed.),(1980), Essays in Multistate Mathematical Demography, IIASA, RR-80-10.

Rogers A.(ed.), (1981), Advances in Multiregional Demography, IIASA, RR-81-6.

Schoen & Land, (1979), "A General Algorithm for Estimating a Markov-generated Increment-decrement Life Tables with Applications to Marital Status Patterns", Journal of the American Statistical Association, 74(1979)761-776.

Schoen R., (1979) "Calculating Increment-decrement Life Tables by Estimating Mean Durations at Transfer from Observed Rates", Math. Biosciences, 47(1979)255-269.

Willekens F. & Rogers A., (1978), Spatial Population Analysis: Methods and Computer Programs, IIASA, RR-78-18.

Willekens et al., Multistate Analysis of Marital Status Life Tables, Theory and Application, WP.17, NIDI, Voorburg.

Land K.C. & Rogers A., (1982), Multiregional Mathematical Demography, Academic Press, N.Y.

Kitsul P. & Philipov D., (1981), "The One-year/five-year Migration Problem", in Rogers A. (ed.), 1981.

Ledent J., (1981), "Constructing Multiregional Life Tables using Place-of-birth-specific Migration Data", in Rogers(ed.), 1981.